Graphical construction of cardinal points from the transference

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Abstract

Usually nodal, principal and focal points are defined independently and thought of as distinct structures with no simple relationship among them. By adopting a holistic approach, in which these three types of cardinal points are treated as particular cases of a larger class of special points, this paper develops a method of constructing the locations of the cardinal points of a system graphically directly from the transference. The method provides a useful way of visualising the relationship of the locations of the cardinal points and of how they are affected by changes in the system or when a second optical system is placed in front of the first. The paper illustrates the graphical procedure by applying it in several situations of interest in optometry and ophthalmology, including the effect of a contact lens or refractive surgery on a reduced eye and the effect of accommodation, a spectacle lens and an afocal telescope on the Gullstrand-Emsley schematic eye. (*S Afr Optom* 2011 **70**(1) 3-13)

Key words: Cardinal point, nodal point, principal point, focal point, transference

Introduction

The purpose of this paper is to show how one can obtain the locations of cardinal and other special points by graphical construction directly from the transference of the dioptric system in question. The graphical construction is illustrated by means of several examples of optometric and ophthalmological interest, namely a reduced eye, a reduced eye that undergoes refractive surgery, a contact lens on the reduced eye, accommodation of the Gullstrand-Emsley schematic eye and a schematic eye with various optical devices in front of it including a thin spectacle lens and a telescope.

A key issue here is the unification of focal, nodal and principal points under the general heading of special points. This leads to a pair of straight lines, called the locator lines, superimposed on a sketch of the system from which one can obtain the positions of the individual cardinal points by construction. The slopes and cutting points of the straight lines come directly from the entries of the system's transference. Changes in an eye or lenses placed in front of the eye change the locator lines and so change the locations of the cardinal points. This provides for holistic visualization of the locations of individual cardinal points and of how they are altered by accommodation, refractive surgery and by contact and spectacle lenses and telescopes in front of the eye.

To begin we turn to the important concept of the transference in Gaussian optics.

A general system in Gaussian optics

Consider an arbitrary dioptric system S as represented in Figure 1(a). It has an *optical axis* Z which defines the general direction of light through S and the positive sense for specifying longitudinal position. It is bounded by two transverse planes T_0 and T_1, the *entrance* and *exit* planes, respectively. The distance $z \geq 0$ from T_0 to T_1 is the *length* of S. Immediately upstream from T_0 (that is, to the left of T_0 in Figure 1) the medium has index of refraction n_0 ; immediately downstream from T_1 (to the right of T_1 in the figure) the index is n_1 . None of the optical elements within S is shown in Figure 1(a).

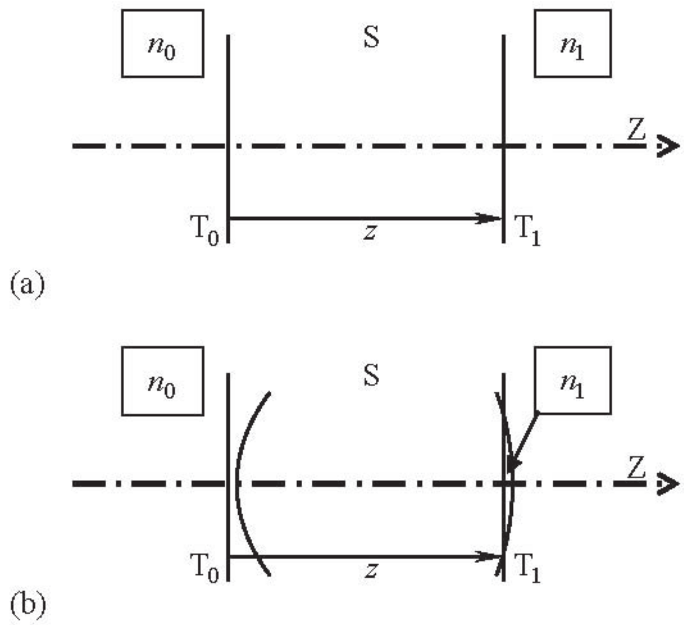

Figure 1 (a) An arbitrary dioptric system S with optical axis Z. None of the optical elements of S is shown. Z defines the positive sense through S. T_0 is the entrance plane, T_1 the exit plane and *z* is the length of S. The indices of refraction immediately upstream of T_0 and immediately downstream of T_1 are n_0 and n_1 respectively. (b) S is the visual system of an eye. T_0 lies immediately upstream of the first surface of the tear film (shown as a curved surface) on the cornea and T_1 immediately upstream of the retina (also shown).

System S may be the visual system of an eye; see Figure 1(b). In that case entrance plane T_0 is immediately in front of K, the first surface of the tear layer on the cornea, and exit plane T_1 is immediately in front of R the retina. The cornea and tear layer are inside visual system S while R is not.

In Gaussian optics the optical character of system S is represented by the 2×2 matrix[1-8]

$$\mathbf{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

where the entries, called the *Gaussian coefficients* of S, are related by

$$AD - BC = 1. \tag{2}$$

We refer to A as the *dilation*, B as the *disjugacy*, C as the *divergence* and D as the *divarication* of system S.[9-11]

The dioptric power F of S is defined by[9]

$$F = -C. \tag{3}$$

It is convenient usually to work in terms of C. Of course, at any stage one can replace C by $-F$.

Cardinal and other special points of a general system

Systems other than afocal systems, that is, systems with $F \neq 0$, have six cardinal points, three associated with incident segments of rays and three with emergent segments. We distinguish them as *incident* and *emergent* cardinal points respectively. (For reasons presented elsewhere[10, 12] we prefer these terms to the corresponding terms, *first* and *second*, respectively, frequently used in the literature.) The cardinal points are located on the optical axis which is represented as Z in the figures.

Several authors list expressions in terms of the Gaussian coefficients for the locations of the six cardinal points.[3, 5, 7] The expressions take the form of six separate equations. A key element in the analysis presented in this paper is the recognition of the fact that the expressions for the locations of the three incident cardinal points can be unified into the single equation

$$z_{Q0} = n_0 (D - X)/C \tag{4}$$

and similarly for the three emergent cardinal points one can write the single equation

$$z_{Q1} = -n_1 (A - 1/X)/C\,. \tag{5}$$

The proviso is that $C \neq 0$. In these equations Q represents a cardinal point. It takes the place of F, P and N for focal, principal and nodal points respectively. Q_0 represents an incident cardinal point and Q_1 an emergent cardinal point. X is a scalar we term the *characteristic* of the cardinal point; its values are listed for each type of cardinal point in Table 1. z_{Q0} is the longitudinal position of incident cardinal point Q_0; it is measured along Z from entrance plane T_0 of system S (see Figure 2). z_{Q1} is the longitudinal position of emergent cardinal point Q_1 and it is measured along Z but from exit plane T_1. (In introducing the characteristic X we are applying in Gaussian optics what has already been done in linear optics[12].)

Table 1 Cardinal points and their characteristics X.

Cardinal point		Symbol	Characteristic X
Incident	Focal	F_0	0
	Principal	P_0	1
	Nodal	N_0	n_1/n_0
	-		∞
Emergent	-		0
	Principal	P_1	1
	Nodal	N_1	n_1/n_0
	Focal	F_1	∞

So far X assumes only those values listed in Table 1. It is useful now to generalize X so that it can assume any value on the real number line, including ∞ (the emergent focal point) and negative numbers (which would allow for other points such as anti-principal and anti-nodal points[12-14] and the points described by Malacara[12, 15, 16]). Characteristic X then defines an infinity of points Q_0 whose locations are given by Equation 4 and another infinity of points Q_1 with locations given by Equation 5. We refer to Q as a *special* point with characteristic X. Cardinal points are special points. Q_0 and Q_1 are incident and emergent special points respectively.

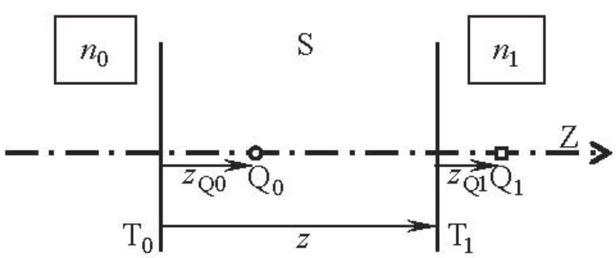

Figure 2 Incident special point Q_0 of system S is located at longitudinal position z_{Q0} relative to entrance plane T_0. Emergent special point Q_1 is located at position z_{Q1} relative to exit plane T_1.

Corresponding to each value of X is a pair of conjugate points, an incident special point Q_0 and an emergent special point Q_1. Note, however, that the focal points F_0 and F_1 are not conjugate.

By allowing X to be any scalar we have converted each of Equations 4 and 5 from three equations (one for each value of X as listed in Table 1) to an infinity of equations. In fact, instead of three equations, they become the equations of straight lines that form the basis of the construction to which we now turn.

(An interpretation of the meaning of the characteristic X of a special point is presented in the Appendix.)

Graphical representation

It is useful to express Equations 4 and 5 as follows

$$X = -Cz_{Q0}/n_0 + D \tag{6}$$

and

$$1/X = Cz_{Q1}/n_1 + A. \tag{7}$$

Equations 6 and 7 are the equations of straight lines. Because they can be used to find the locations of the special points, including the cardinal points in particular, we term them the *locator lines* of the system. Provided n_0 and n_1 are known it is clear that the equations for the locator lines can be written down directly from the transference of the system. In fact they exist uniquely for any system, including afocal systems ($C = 0$).

The graphical procedure is illustrated in Figure 3 and described as follows: Axis X is superimposed on entrance plane T_0 and axis $1/X$ is superimposed on exit plane T_1 with 0 for both axes at optical axis Z. Points below Z represent negative characteristics. Locator lines L_0 and L_1, are constructed from the entries of the transference **S** according to Equations 6 and 7 respectively, L_0 from the bottom row and L_1 from the first column of **S**. Incident locator line L_0 has slope $-C/n_0$ and intersection $X = D$ in T_0; emergent locator line L_1 has slope C/n_1 and intersection $1/X = A$ in T_1.

The ratio of the slope of L_1 to the slope of L_0 is $-n_0/n_1$ for any system. If L_0 has positive slope (the case illustrated in Figure 3) then L_1 has negative slope and vice versa.

To find the location of incident special point Q_0 of characteristic X one constructs a line parallel to Z at that value of X in T_0. From the intersection of that line with L_0 a line is dropped perpendicular to Z. Q_0 is where this last line intersects Z. The corresponding emergent special point Q_1 is found similarly but starting with the

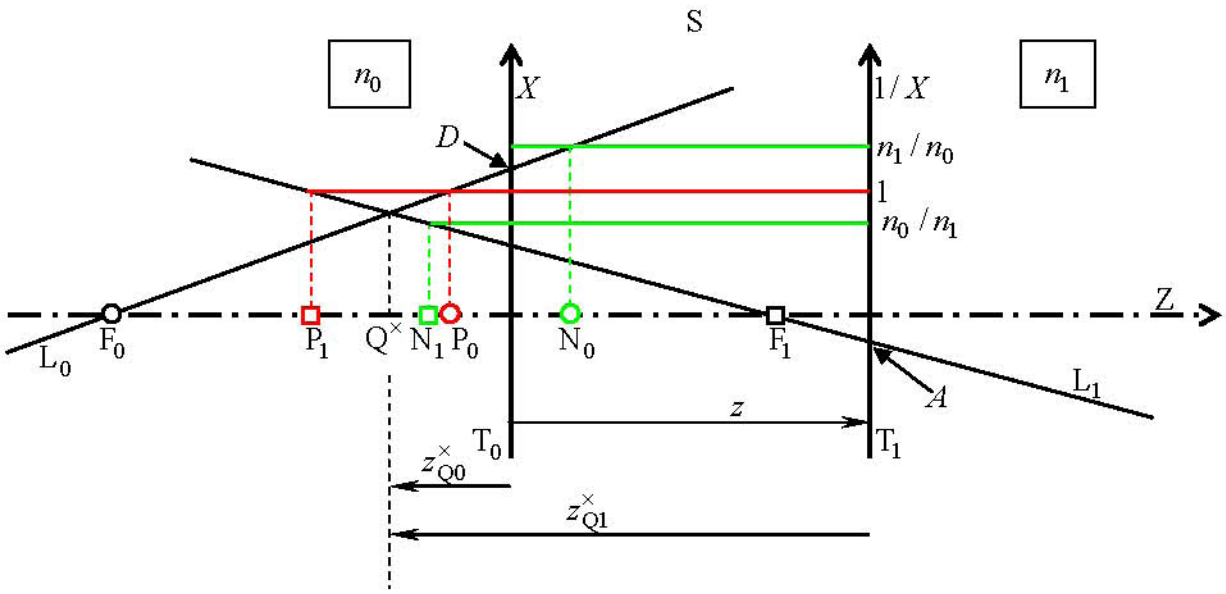

Figure 3 The locator lines L_0 and L_1 for system S. They represent Equations 6 and 7 respectively. Axis X is superimposed on entrance plane T_0 and axis $1/X$ on exit plane T_1. L_0 intersects T_0 in D and the optical axis Z at the incident focal point F_0. L_1 intersects T_1 in A and Z at the emergent focal point F_1. The principal (red) and nodal (green) points are shown. The locator lines intersect in a point corresponding to $\mathrm{Q}^\times$ on Z with longitudinal position given by Equation 8. Incident and emergent points are represented by means of circles and squares respectively.

value of $1/X$ in T_1. The reader may wish to use this procedure to check the locations of the six cardinal points of the system in Figure 3. $X = 1 = 1/X$ locates principal points P_0 and P_1 (shown red). At $X = n_1/n_0$ and $1/X = n_0/n_1$ the locator lines locate nodal points N_0 and N_1 (green). The incident focal point F_0 is located at the intersection of L_0 with the optical axis Z (where $X = 0$) and the emergent focal point F_1 at the intersection of L_1 with Z.

For the visual system of the eye $C < 0$ and $n_1 > n_0$. In such cases L_0 has positive slope and L_1 negative slope of lower magnitude. Thus incident special points occur along Z in the order of increasing characteristic X and the same is true for the emerge nt special points. That means, in particular, that the incident cardinal points always have the order F_0, P_0, N_0 and the emergent cardinal points the order P_1, N_1, F_1. Without additional information we can say nothing about the positions of the emergent points relative to the incident points.

The higher the power F of the system the steeper the incident locator line L_0 and, hence, the closer together are the incident cardinal points, the same being true of the emergent cardinal points. (We are referring here to the magnitude of steepness and we are using 'steep' in the conventional sense of slope rather than in the bizarre sense of curvature frequently encountered in the contact lens literature.)

Corresponding to the intersection of locator lines L_0 and L_1 is the point $\mathrm{Q}^\times$ on Z. At the intersection the two lines have the same vertical coordinates so one can equate the right-hand sides of Equations 6 and 7. Their horizontal positions are also the same but $z^\times_{\mathrm{Q}0}$ is measured relative to T_0 and $z^\times_{\mathrm{Q}1}$ to T_1 a distance z from T_0. Hence we can equate $z^\times_{\mathrm{Q}0}$ and $z^\times_{\mathrm{Q}1} + z$ (see Figure 3). Solving the two equations we find that $\mathrm{Q}^\times$ has longitudinal position

$$z^\times_{\mathrm{Q}0} = \frac{D - A + Cz/n_1}{C(1/n_1 + 1/n_0)} \tag{8}$$

relative to T_0.

Altered systems and shifts of special points

The results above apply to the visual system of the eye in particular. They also apply to the combination of spectacle lens and visual system when a spectacle lens is placed in front of an eye. Typically we shall add a subscript C to distinguish symbols representing the combination from those without a C which will

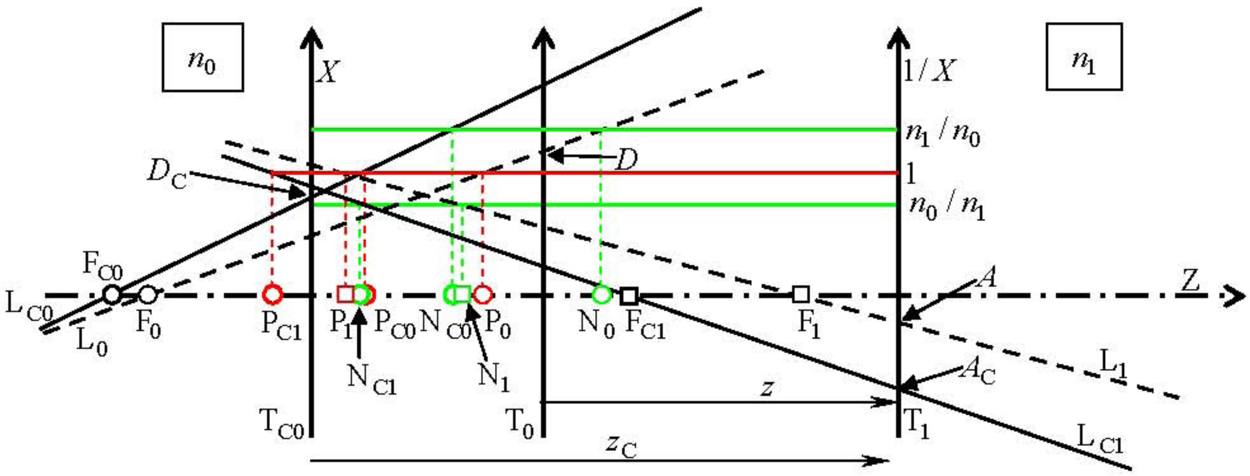

Figure 4 Combined system S_C of length z_C and with entrance plane T_{C0} and exit plane T_1. It is the visual system S of Figure 3 with an optical device in front of it. Locator lines L_0 and L_1 of S become L_{C0} and L_{C1} of S_C. Cardinal points F_0, P_0, N_0, P_1, N_1 and F_1 shift to F_{C0}, P_{C0}, N_{C0}, P_{C1}, N_{C1} and F_{C1} .

usually represent the visual system. Subscript C will also be used for combinations of other optical devices and visual systems and for visual systems following refractive surgery, intra-ocular lens implantation or accommodation. Thus S will usually represent the visual system of an (unaccommodated) eye and S_C the combination of a lens and the visual system or the visual system of an eye altered by accommodation or surgery. Corresponding to each of S and S_C will be a graph like that of Figure 3. We shall be interested in how the two compare.

Figure 4 repeats Figure 3 for a visual system S and includes the construction for the combination S_C of visual system with spectacle lens in front of it. The figure allows one to compare the locations of the cardinal points in the two cases.

For combined system S_C Equation 4 becomes

$$z_{QC0} = n_0 (D_C - X)/C_C . \tag{9}$$

But this is the position relative to entrance plane T_{C0} of S_C. Let S_C have length z_C. T_{C0} is a distance $z_C - z$ upstream from T_0. It follows that incident special points undergo shifts of

$$\Delta z_{Q0} = z_{QC0} - (z_C - z) - z_{Q0} \tag{10}$$

or

$$\Delta z_{Q0} = -n_0 X (1/C_C - 1/C) + n_0 (D_C / C_C - D/C) - (z_C - z). \tag{11}$$

Similarly one finds that the emergent special points shift by

$$\Delta z_{Q1} = n_1 (1/C_C - 1/C)/X - n_1 (A_C / C_C - A/C). \tag{12}$$

Figure 5 is a schematic representation of locations and shifts of special points.

One notes from Equation 11 that all incident special points undergo the same shift if and only if $C_C = C$. The same is true of emergent special points as is clear from Equation 12.

So far our discussion has been general. We turn now to some examples selected for their relevance to optometry and ophthalmology. We begin with what is, perhaps, the simplest system of interest, the reduced eye.

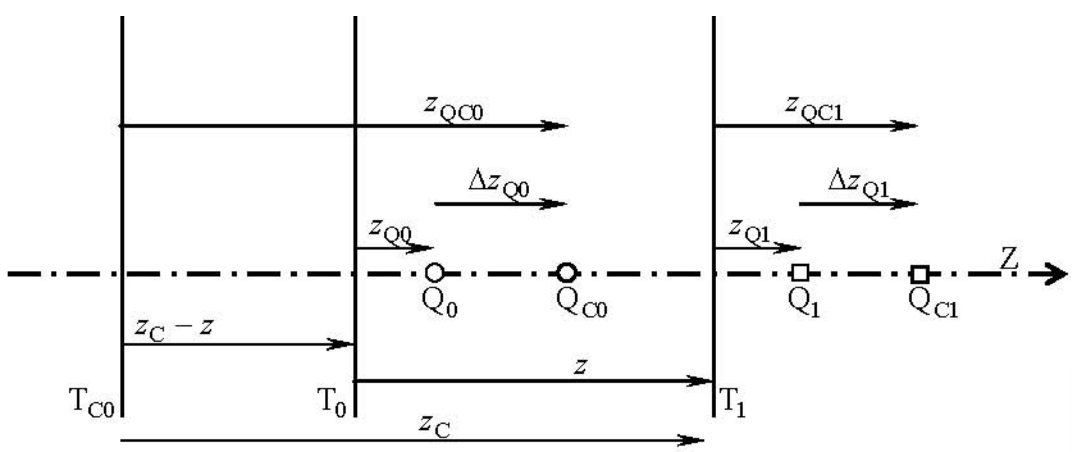

Figure 5 Representation of the positions and relative positions of incident and emergent special points for system S (Q_0 and Q_1) and for combined system S_C (Q_{C0} and Q_{C1}).

Reduced eye

Consider now a single refracting surface (the 'cornea') of power F_K followed by a homogeneous gap of width z and index n_1. Following the usual procedure of writing transferences for the two elementary systems and multiplying them in reverse order we find that the visual system of the reduced eye has transference

$$\mathbf{S} = \begin{pmatrix} 1 & z/n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -F_K & 1 \end{pmatrix},$$

that is

$$\mathbf{S} = \begin{pmatrix} 1 - zF_K/n_1 & z/n_1 \\ -F_K & 1 \end{pmatrix}. \tag{13}$$

By Equation 1 this defines the Gaussian coefficients of the reduced eye. In particular the ametropia of the eye is[17]

$$A = 1 - F_K z / n_1 \,. \tag{14}$$

The eye's corneal-plane refraction is[17]

$$F_0 = A/B\,, \tag{15}$$

that is,

$$F_0 = n_1/z - F_K \,. \tag{16}$$

From the transference and Equation 6 we see that locator line L_0 has slope F_K/n_0. (In most cases n_0 is 1 and can be discarded; we shall retain the symbol, however, in the interests of generality.) Also from the transference we see that L_0 intersects entrance plane T_0 at $D = 1$. Substitution for A, C and D from Equation 13 into Equation 8 we find that $z_{Q0}^{\times} = 0$ from which we conclude that the two locator lines intersect in T_0. The complete picture is as sketched in Figure 6.

We note that the principal points coincide and that they lie in the entrance plane T_0; they are shown as point P. The nodal points also coincide (N). These observations agree with representations often presented in the literature[18, 19]. (In fact the principal and nodal points are the only special pairs that do coincide as can be shown by making use of Equations 4 and 5; one obtains a quadratic equation in X of which the roots are 1 and n_1/n_0.)

Because the emergent focal point F_1 lies behind the exit plane T_1 the eye of Figure 6 happens to be hyperopic.

Reduced eye and contact lens or altered reduced eye

Suppose now that a thin contact lens of power F_L is placed on the simple eye described above and that the tear lens is negligible. We can lump the power of the contact lens and cornea together and treat them as equivalent to a single refracting surface. The effect is to change the slopes of the locator lines without changing their intersection. L_0 becomes L_{C0} and L_1 becomes L_{C1} as illustrated in Figure 6. (The figure illustrates the case with $F_L > 0$.) L_0 rotates anticlockwise about the point of intersection increasing in slope from F_K/n_0 to $(F_L + F_K)/n_0$. L_1 rotates clockwise about the same point.

It follows that the principal points are not displaced; they remain on the entrance plane T_0. All other special points, including the other cardinal points, move toward T_0. Incident special points with $X > 1$ move negatively (to the left in Figure 6) in proportion to their distance from T_0; incident special points with $X < 1$, including the incident focal point F_0, move positively. More particularly Figure 6 represents the case in which the contact lens compensates for the refraction; the emergent focal point F_1 moves negatively onto the exit plane T_1 and, hence, onto the retina.

In the case of a thin contact lens of negative power the locator lines pivot in the opposite sense shifting the cardinal points in the reverse direction by amounts in proportion to their distance from the entrance plane.

Explicit expressions for the shifts of the special points,

$$\Delta z_{Q0} = -n_0 \left(\frac{1}{F_K} - \frac{1}{F_K + F_L} \right)(X - 1) \tag{17}$$

and

$$\Delta z_{Q1} = n_1 \left(\frac{1}{F_K} - \frac{1}{F_K + F_L} \right)\left(\frac{1}{X} - 1 \right), \tag{18}$$

can be obtained from Equations 11 and 12.

Differentiating Equations 17 and 18 with respect to F_L and using typical values ($F_K = 60$ D, $n_0 = 1$ and $n_1 = 1.3333$) one obtains

Table 2 Approximate shifts in location of the cardinal points caused by a contact lens on a reduced eye.

Cardinal point	Shift in mm/D
F_0	0.28
P	0
N	–0.06
F_1	–0.37

$$\frac{\partial \Delta z_{Q0}}{\partial F_L} \approx -\frac{5}{18}(X-1) \text{ mm/D} \quad (19)$$

and

$$\frac{\partial \Delta z_{Q1}}{\partial F_L} \approx \frac{10}{27}\left(\frac{1}{X}-1\right) \text{ mm/D} \quad (20)$$

for F_L of small magnitude. Table 2 lists the approximate shifts of the cardinal points in mm/D for a contact lens. The figure for the shift of emergent focal point F_1 agrees with the figure presented by Bennett and Rabbetts and the 3/8 mm/D sometimes cited[20].

Assuming that the only change to the eye is to the power of the cornea the results above can also be applied to refractive surgery. In that case F_L is the increase in power of the cornea resulting from the surgery.

Accommodation

We examine the Gullstrand-Emsley schematic eye, details of which are given by Bennett and Rabbetts[20]. Symbolically the relaxed eye can be represented as $|42.73[3.60/1.3333]|8.27[3.60/1.4160]|13.78[16.69/1.3333]$.

This should be read in sequence as a refracting surface of power 42.73 D, a homogeneous gap of width 3.60 mm and index 1.3333, a refracting surface of power 8.27 D, and so on. The accommodated schematic eye is $|42.73[3.20/1.3333]|16.54[4.00/1.4160]|16.54[16.69/1.3333]$.

Multiplying the transferences of the elementary systems in reverse order as usual one obtains the transferences

$$\mathbf{S} = \begin{pmatrix} 0.0003 & 16.53\ \text{mm} \\ -0.0605\,\text{kD} & 0.9062 \end{pmatrix} \quad (21)$$

and

$$\mathbf{S}_C = \begin{pmatrix} -0.1380 & 16.08\ \text{mm} \\ -0.0697\,\text{kD} & 0.8757 \end{pmatrix} \quad (22)$$

for visual systems S and S_C of the relaxed and accommodated eyes respectively.

From Equations 21 and 22 we see that the power increases from about 60.5 D to about 69.7 D. The divarication decreases from 0.9062 to 0.8757 and the

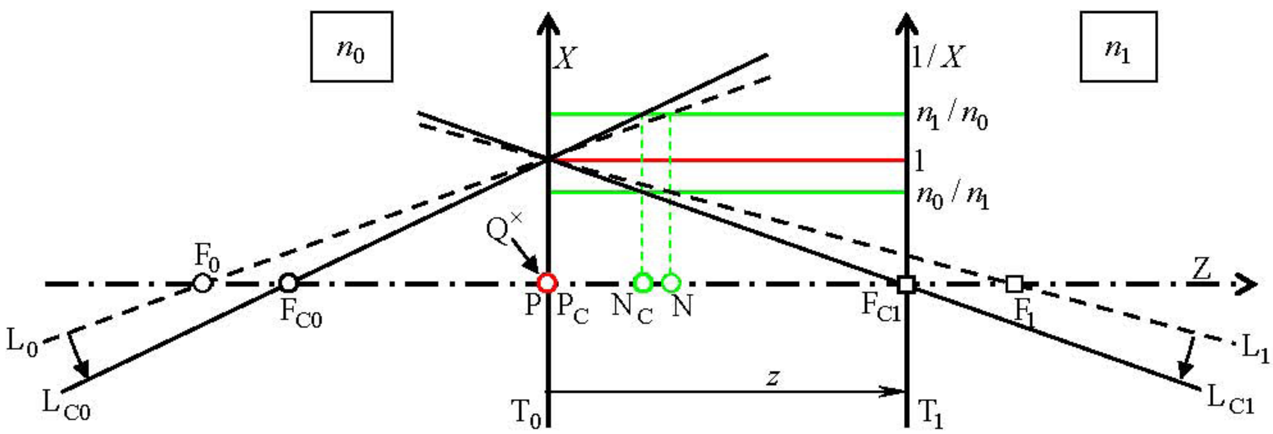

Figure 6 Locator lines and cardinal points for S, the visual system (L_0, L_1 , F_0 , P, N and F_1), and for S_C, the combination of thin contact lens and visual system (L_{C0}, L_{C1}, F_{C0}, P_C, N_C and F_{C1}). In this case the eye is hyperopic and the contact lens compensates for the refractive error. The effect of the lens is to cause the locator lines to rotate about $X=1$ on entrance plane T_0 . The principal points remain fixed while other cardinal points and, in fact, all other special points, shift towards T_0 in proportion to their distances from T_0.

dilation decreases from 0.0003 to -0.1380 . It follows that the locator lines steepen and that the intersections of the locator lines with the entrance and exit planes decrease. The construction is shown in Figure 7.

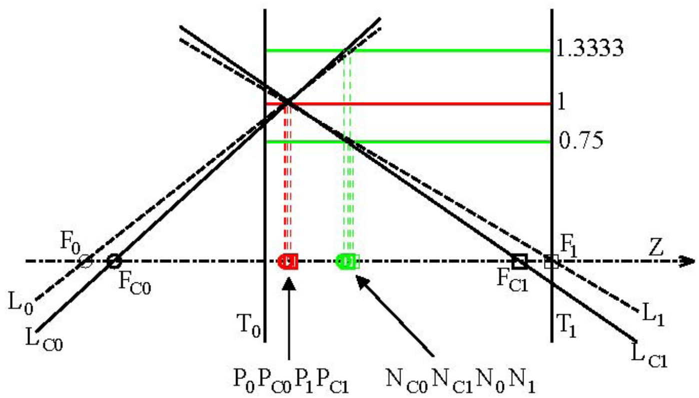

Figure 7 The locator lines and cardinal points of the visual system of the relaxed S and accommodated S_C Gullstrand-Emsley schematic eye. There is an approximate pivot point at about 2 mm from T_0. The principal points are located at about this longitudinal position; they are little affected by accommodation. Accommodation steepens the locator lines and, approximately speaking, draws all other special points towards the principal points.

In contrast to the case with the reduced eye (Figure 6) the incident and emergent principal points now dissociate although only slightly. As for the reduced eye the increase in the power of the eye steepens the locator lines but now the locator lines do not have a common point of intersection on the entrance plane T_0. The point of intersection is at about 1.7 mm downstream from T_0 for the relaxed eye and 1.9 mm for the accommodated eye. This also defines the approximate position of the principal points which undergo little displacement as a result of accommodation. Accommodation draws all the other special points towards the principal points. The principal points are in the order $P_0\,P_{C0}\,P_1\,P_{C1}$ and the nodal points in the order $N_{C0}\,N_{C1}\,N_0\,N_1$.

Spectacle lens

Suppose a thin lens of power 3 D is placed 15 mm in front of the Gullstrand-Emsley schematic eye. One obtains the transference

$$\mathbf{S}_\mathrm{C} = \begin{pmatrix} -0.0493 & 16.53\text{ mm} \\ -0.0605\text{ kD} & -0.0010 \end{pmatrix} \qquad (23)$$

for the compound system S_C of spectacle lens and visual system. Comparison of Equations 21 and 23 shows that the powers of S and S_C are the same at least to the number of decimal places shown. Consequently the slopes of the locator lines are unchanged. In fact, as shown in Figure 8, the incident locator line hardly changes which means that the incident cardinal points hardly move.

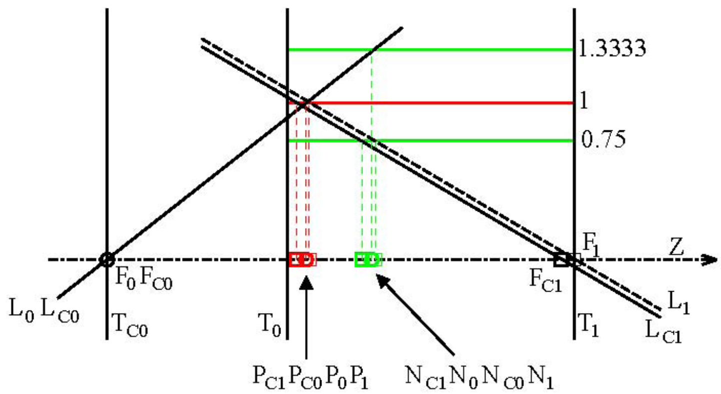

Figure 8 Locator lines and cardinal points for S, the visual system of the relaxed Gullstrand-Emsley eye, and for S_C, system S with a thin lens of power 3 D at 15 mm in front of it. There is almost no change in incident locator and, hence, the incident special points undergo almost no shift. The emergent locator moves negatively with almost no change in slope and, hence, the emergent special points undergo a nearly uniform negative shift.

Because the emergent locator line moves negatively without changing slope all the emergent special points undergo the same negative displacement; there is uniform negative displacement of the emergent special points.

Repeating the calculation for spectacle lenses of other powers shows that the emergent points move negatively by about 0.36 mm per dioptre of the power of the spectacle lens.

Eye and telescope

Consider an afocal telescope with transference

$$\mathbf{S}_\mathrm{T} = \begin{pmatrix} 1/D_\mathrm{T} & B_\mathrm{T} \\ 0 & D_\mathrm{T} \end{pmatrix} \qquad (24)$$

in front of an eye. (D_T is usually called the magnification of the telescope and represented by the symbol M.) The combined system of eye and telescope has transference

$$\mathbf{S}_C = \begin{pmatrix} A/D_T & AB_T + BD_T \\ C/D_T & CB_T + DD_T \end{pmatrix}. \tag{25}$$

From the divergence one immediately sees that a telescope divides the slopes of the locator lines by the magnification. For $D_T > 1$ this causes the incident special points, and the emergent special points to spread out in proportion to the magnification.

Figure 9 shows the effect on the special points of the particular telescope $|20[25]|-40$ on the special points of the visual system of the Gullstrand-Emsley schematic eye. The telescope has transference

$$\mathbf{S}_T = \begin{pmatrix} 0.5 & 25\text{ mm} \\ 0\text{ kD} & 2 \end{pmatrix}$$

and, hence, magnification 2. The combined system has transference

$$\mathbf{S}_C = \begin{pmatrix} 0.0001 & 33.06\text{ mm} \\ -0.0302\text{ kD} & 0.3004 \end{pmatrix}. \tag{26}$$

The entrance plane T_{C0} of the combined system is off the figure to the left and so is the incident focal point F_{C0}. The magnitudes of the slopes of the locator lines are halved. The emergent locator line L_1 pivots about the emergent focal point F_1 on the retina. It has the effect of pushing the special points away in the negative sense and spreading them out. The incident locator line L_0 effectively pushes the incident nodal point N_0 closer to the retina (becoming N_{C0}) but pulls the incident principal point P_0 in the opposite direction. Notice how the nodal points N_0 and N_1, initially close together, get pulled apart and have their order reversed. The same happens to the principal points P_0 and P_1. (P_{C1} is off the figure to the left.)

For $D_T < -1$ the telescope is Keplerian (images are inverted), a device that changes the sign of the slopes of the locator lines.

For reverse telescopes ($-1 < D_T < 1$ and images are minified) the locator lines steepen drawing the incident points, and the emergent points, closer together.

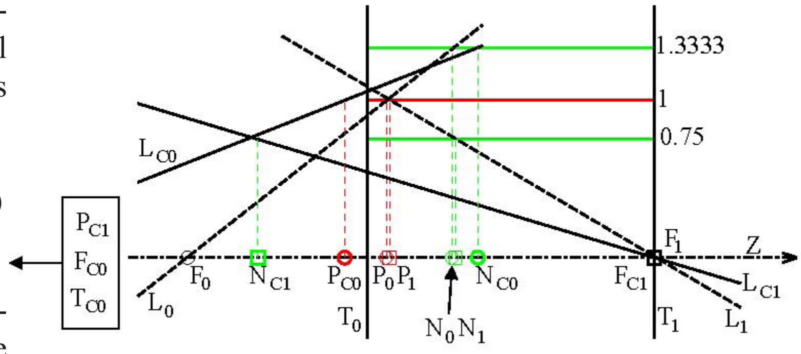

Figure 9 Effect of an afocal telescope of magnification 2 in front of the Gullstrand-Emsley schematic eye. The entrance plane of the telescope T_{C0} is off the figure to the left as are the incident focal point F_{C0} and the emergent principal point P_{C1} of the combined system of telescope and visual system. The horizontal red and green lines have been drawn only between T_0 and T_1 .

Compensated visual system

No matter what system is in front of the eye if distant points map to points on the retina then the combined system S_C has $A_C = 0$ [17] and Equation 5 reduces to

$$z_{QC1} = \frac{n_1}{XC_C}. \tag{27}$$

Thus the location of the emergent special points depends only on the divergence of the combined system. If the device is an afocal telescope then

$$z_{QC1} = \frac{n_1 D_T}{XC}, \tag{28}$$

that is, emergent special points are pushed away from the retina in proportion to the magnification.

The location of the incident special points depends on both C_C and D_C.

Concluding remarks

By departing from common practice and viewing cardinal points, not as unrelated entities, but as particular special points we are led to a graphical construction which gives insight into the positions of cardinal points and how their locations are affected by changes to the system and by placing other systems in front of the system in question. The construction depends on the concept of locator lines whose structures are defined directly by the transferences of the systems. Depending on the characteristic X the locator lines locate the special points. The locator lines provide a means of visualizing the locations of the cardinal points and the effects on them of changes to the system.

The slopes of the locator lines are proportional to the divergence C (or power F) of the system. The greater the magnitude of the divergence the closer together are the incident special points and the emergent special points. The incident locator line intersects the entrance plane of the system in the divarication D of the system and the emergent locator line intersects the exit plane in the dilation A.

Several examples have been presented of situations of relevance for optometry and ophthalmology. When applied to the visual system A is the ametropia of the eye. It is the effective ametropia when there is a spectacle lens or other optical device in front of the eye.

For a reduced eye the incident and emergent principal points coincide and are located on the entrance plane. The incident and emergent nodal points also coincide. Refractive surgery that changes the power of the cornea of the reduced eye leaves the location of the principal points unchanged. The rest of the special points are drawn towards the principal points if the power of the cornea is increased as represented in Figure 6. They are pushed away from the principal points if the corneal power is reduced. A thin contact lens on the reduced eye has the same effect.

A more realistic eye is the Gullstrand-Emsley schematic eye; it is represented in Figure 7. The two principal points are now separate but still close together. They are located a little less than 2 mm after the entrance plane of the eye. The nodal points are also separate. Nevertheless there are similarities with the reduced eye (compare Figures 6 and 7). Accompanying accommodation of the schematic eye the locator lines rotate about an approximate pivot point effectively pushing the principal points a little closer together and towards the principal points.

It can happen that placing an optical system in front of the visual system has no effect on the power. This happens, at least approximately, in the case of a thin spectacle lens (Figure 8). The slope of the locator lines then does not change. There is then uniform displacement of the emergent special points. There is also uniform displacement of the incident special points but the magnitude of the displacement is apparently much less. In fact in the case represented by Figure 8 the incident special points remain essentially unshifted. (Related to these matters is Pascal's 'benzene ring'[21-23], a topic dealt with in a recent paper[24].)

An afocal telescope has a more dramatic effect on the locations of the cardinal points (see Figure 9). The slopes of the locator lines are divided by the magnification of the telescope. This causes the special points to spread out in the case of the typical (Galilean) telescope.

In this paper we have made use of several applications for illustrative purposes. The intention, however, has not been to give an exhaustive treatment of those applications; indeed one expects that each of the applications could well warrant further exploration in its own right.

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Appendix

The meaning of the characteristic X of special points

For a system S with transference S one has[4]

$$Cy_0 + D\alpha_0 = \alpha_1 \cdot \tag{A1}$$

A ray incident onto the entrance plane at transverse position y_0 and with reduced inclination α_0 emerges from the exit plane at transverse position y_1 and with reduced inclination α_1.

Consider the compound system S_C from the plane of an incident special point Q_0 to the plane of the conjugate emergent special point Q_1. It has transference

$$\mathbf{T}_C = \begin{pmatrix} 1 & z_{Q1}/n_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -z_{Q0}/n_0 \\ 0 & 1 \end{pmatrix}$$

which multiplies out to give

$$D_C = -Cz_{Q0}/n_0 + D \cdot$$

Substituting for z_{Q0} from Equation 4 we obtain

$$D_C = X$$

where X is the characteristic of the two special points. Consider now a ray entering S_C at z_{Q0}. It has $y_0 = 0$. Substituting into Equation A1 one obtains

$$X\alpha_0 = \alpha_1 .$$

This means that the characteristic X of a special point is the factor by which the reduced inclination of rays incident onto the point are magnified across the system.

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