Aperture referral in dioptric systems with stigmatic elements

WF Harris

Department of Optometry, University of Johannesburg, PO Box 524, Auckland Park, 2006 South Africa

Received 11 July 2011; revised version accepted 3 February 2012

<wharris@uj.ac.za>

Abstract

A previous paper develops the general theory of aperture referral in linear optics and shows how several ostensibly distinct concepts, including the blur patch on the retina, the effective corneal patch, the projective field and the field of view, are now unified as particular applications of the general theory. The theory allows for astigmatism and heterocentricity. Symplecticity and the generality of the approach, however, make it difficult to gain insight and mean that the material is not accessible to readers unfamiliar with matrices and linear algebra. The purpose of this paper is to examine what is, perhaps, the most important special case, that in which astigmatism is ignored. Symplecticity and, hence, the mathematics become greatly simplified. The mathematics reduces largely to elementary vector algebra and, in some places, simple scalar algebra and yet retains the mathematical form of the general approach. As a result the paper allows insight into and provides a stepping stone to the general theory. Under referral an aperture undergoes simple scalar magnification and transverse translation. The paper pays particular attention to referral to transverse planes in the neighbourhood of a focal point where the magnification may be positive, zero or negative. Circular apertures are treated as special cases of elliptical apertures and the meaning of referred apertures of negative radius is explained briefly. (*S Afr Optom* 2012 **71**(1) 3-11)

Key Words: aperture referral, stigmatic systems, blur patch, effective corneal patch, field of view, projective field.

Introduction

The blur patch on the retina, the effective patch on the cornea, or on any refracting surface, the projective field of a retinal point and the field of view of an optical instrument, all important in vision, are usually thought of as distinct concepts that need to be treated separately. A recent paper[1] shows, however, that each is a special case of a general phenomenon described there as *aperture referral*. The concepts represent the action of an aperture, often the pupil of the eye, taking place as it were at some other axial position in the system. The paper[1] uses linear optics to develop a general theory of aperture referral for general dioptric systems whose refracting elements may be heterocentric and astigmatic.

Allowance for astigmatism in the general theory means that symplecticity takes its full form. Consequently the mathematics is complicated, qualitative insight is difficult to obtain and many readers find the material inaccessible. Here we treat the special case in which all refracting elements are stigmatic. In other words we exclude astigmatism. The exclusion of astigmatism greatly simplifies the mathematics but retains much of the essential mathematical structure. In effect, because of symplecticity, the transference of an optical system simplifies from 20 numbers related by six equations to only eight related by only one equation. (Results concerning simplecticity in the context of linear optics are summarized elsewhere[2].) What is a problem in linear algebra becomes, more or less, a problem in the much more familiar vector algebra. It becomes much easier to visualize what is happening and the special theory provides a stepping stone to the general theory. Thus it gives insight into the general theory that one cannot get in any other way. The purpose of this paper, then, is to examine aperture referral in dioptric systems containing only stigmatic refracting elements. The surfaces may be decentred or tilted. We shall also allow apertures to be elliptical and decentred. Circular apertures are treated as a special case.

Transferences of systems with stigmatic elements

An optical system S has transference (5×5)

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{e} \\ \mathbf{C} & \mathbf{D} & \boldsymbol{\pi} \\ \mathbf{o}^{\mathrm{T}} & \mathbf{o}^{\mathrm{T}} & 1 \end{pmatrix} \tag{1}$$

where $\mathbf{A}$ (the dilation), $\mathbf{B}$ (the disjugacy), $\mathbf{C}$ (the divergence) and $\mathbf{D}$ (the divarication) are the four 2×2 fundamental properties of S, $\mathbf{e}$ (the transverse translation) and $\boldsymbol{\pi}$ (the deflectance) are the two 2×1 fundamental properties and $\mathbf{o}^{\mathrm{T}}$ is the matrix transpose of the 2×1 null matrix $\mathbf{o}$. $\mathbf{e}$ and $\boldsymbol{\pi}$ account for decentration or tilt of refracting surfaces in the system. Together the six fundamental properties represent 20 numbers. The top-left submatrix of 2×2 fundamental properties is symplectic; results for such matrices are summarized elsewhere[2]. For more information on the transference the reader is referred to previous papers[1, 3].

If the longitudinal axis coincides with the optical axis[4] of the system $\mathbf{e}$ and $\boldsymbol{\pi}$ are null and one can eliminate the fifth row and column and reduce the transference to 4×4.

By a stigmatic refracting element we mean either a refracting surface that is not astigmatic (it is invariant under rotation about a normal) or a gap of material that is homogeneous and isotropic. We make the following assertion: every dioptric system all of whose elements are stigmatic has transference

$$\mathbf{T} = \begin{pmatrix} \mathbf{I}A & \mathbf{I}B & \mathbf{e} \\ \mathbf{I}C & \mathbf{I}D & \boldsymbol{\pi} \\ \mathbf{o}^{\mathrm{T}} & \mathbf{o}^{\mathrm{T}} & 1 \end{pmatrix} \tag{2}$$

where $\mathbf{I}$ is the 2×2 identity matrix. A, B, C and D are scalars; we shall call them the four scalar fundamental properties of the system. Thus for a system with only stigmatic refracting elements the dilation $\mathbf{A}$ is a scalar matrix $\mathbf{I}A$. We shall also call A the dilation. The same holds for the other 2×2 fundamental properties. Because of symplecticity the scalar fundamental properties are related by

$$AD - BC = 1 . \tag{3}$$

A, B, C, D, $\mathbf{e}$ and $\boldsymbol{\pi}$ make up the eight numbers referred to in the Introduction above and Equation 3 represents the single equation that relates them.

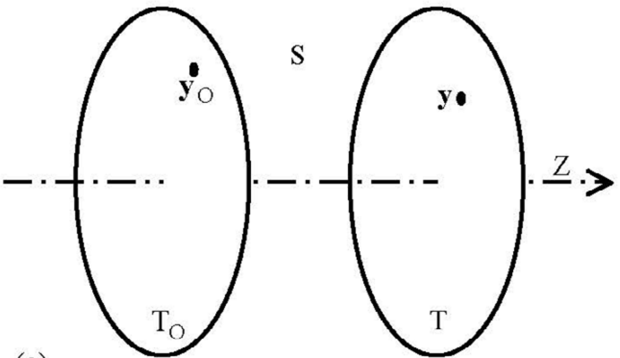
The assertion made above is obviously true for refracting elements: a stigmatic refracting surface has A=D=1, B=0 and C equal to the negative of the (spherical) power F and a homogeneous gap has A=D=1, C=0 and B equal to the reduced length of the gap. Consider now any compound system made up of refracting elements. Its transference can be obtained by multiplying the transferences of the elements. But multiplication of any two matrices of the form in Equation 1 results in a matrix of the same form as can be readily confirmed. This proves the assertion.

We note in passing that the set of all transferences of the form of Equation 2 satisfies all four axioms of a *group* in mathematics: closure under multiplication as we have just seen; associativity (because $(\mathbf{T}_1\mathbf{T}_2)\mathbf{T}_3 = \mathbf{T}_1(\mathbf{T}_2\mathbf{T}_3)$); existence of an identity (the 5×5 identity matrix, which is the transference of a homogeneous gap of zero width and of a flat refracting surface orthogonal to the longitudinal axis); and existence of an inverse $\mathbf{T}^{-1}$ corresponding to every $\mathbf{T}$ in the set. We may call this set the *group of transferences of systems with stigmatic elements*. (The theory of groups is treated in several good texts[5-7]. Groups are the generic as it were of which transferences are a specific. It is important, the author believes, to recognize such relationships because it sets what one is doing within a mathematical context and counteracts isolationist tendencies. It helps to increase communication with other disciplines and the possibility of being able to draw on methods developed elsewhere. It also emphasizes that we are not developing new mathematics but merely applying it.)

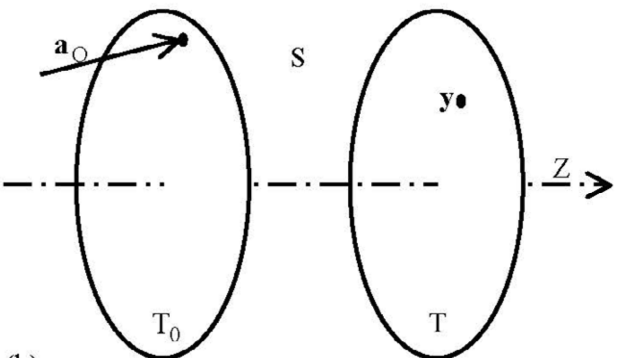
The key to aperture referral is a quantity that is constant along a ray through a system. There are in fact two invariants along a ray, a topic to which we now turn.

Invariants along a ray through a system

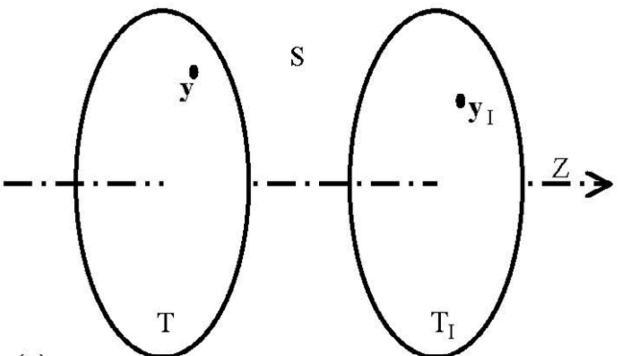
Figure 1 represents four dioptric systems S each with a transference $\mathbf{T}$ of the form of Equation 2. In each case Z is the longitudinal axis; it defines the positive sense through the system. It represents the general direction in which light travels through S. No component of S is shown apart from the entrance and exit planes labelled T with or without a subscript. In each case the index of refraction is n_0 on the negative side of the entrance plane (to the left in Figure 1) and n_1 on the positive side of the exit plane. In (a) and (b) there is an object point O and in (c) and (d) an image point I.



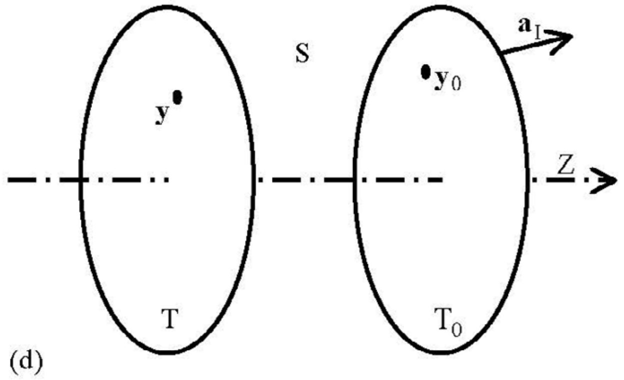
(a)



(b)



(c)



(d)

Figure 1 Longitudinal axis Z and entrance and exit planes (T with or without a subscript) of four dioptric systems S used in the text. None of the elements of the systems is shown. In (a) object point O has transverse position $\mathbf{y}_O$ with respect to Z and is located on the entrance plane T_O of S. A ray (not shown) from O emerges from S at a point with transverse position $\mathbf{y}$ on exit plane T. In (b) a segment of a ray with inclination $\mathbf{a}_O$ and from a distant object point O is incident onto the entrance plane T_0 of S. In (c) a ray is incident onto S at transverse position $\mathbf{y}$ and emerges at image point I at position $\mathbf{y}_I$ on exit plane T_I. In (d) a ray incident at $\mathbf{y}$ emerges at $\mathbf{y}_0$ with inclination $\mathbf{a}_I$ on its way to a distant image point I.

In Figure 1(a) the entrance plane is T_O; it is also the transverse plane containing object point O whose transverse position relative to Z is defined by the position vector $\mathbf{y}_O$. Thus O lies on the entrance plane of system S. Now consider a ray from O through S. At incidence onto S it has inclination $\mathbf{a}_O$. The ray intersects the exit plane T in the point with position vector $\mathbf{y}$ and emerges with inclination $\mathbf{a}$.

Using the transference of Equation 2 we write the two basic equations of linear optics across system S (Equations 1 and 2 of the previous paper[1]) and solve each for $\boldsymbol{\alpha}_O$ where $\boldsymbol{\alpha}_O = n_0\mathbf{a}_O$. One obtains

$$\boldsymbol{\alpha}_O = \left(\mathbf{y} - A\mathbf{y}_O - \mathbf{e}\right)/B \tag{4}$$

and

$$\boldsymbol{\alpha}_O = \left(\boldsymbol{\alpha} - C\mathbf{y}_O - \boldsymbol{\pi}\right)/D \tag{5}$$

where $\boldsymbol{\alpha} = n_1\mathbf{a}$. $\boldsymbol{\alpha}_O$ and $\boldsymbol{\alpha}$ are the reduced inclinations of the ray at incidence and emergence from S.

Now we imagine moving exit plane T to other longitudinal positions while keeping the ray and refracting elements fixed. $\boldsymbol{\alpha}_O$, the left-hand side of both Equations 4 and 5, remains fixed but all of the quantities on the right-hand sides of the two equations vary. Thus, although all the fundamental properties (A, B, C, D, $\mathbf{e}$ and $\boldsymbol{\pi}$) of S vary, as does the emergent state ($\mathbf{y}$ and $\boldsymbol{\alpha}$) of the ray, they change in such a way that the right-hand sides of Equations 4 and 5 remain fixed (because they are each equal to the fixed reduced incident inclination $\boldsymbol{\alpha}_O$). The right-hand sides, therefore, are *invariants* along the ray.

Equations 4 and 5 are Equations 3 and 4 of the previous paper[1] but specialized for systems consisting of elements that are stigmatic. What were equations in vectors and matrices are now equations in vectors and scalars.

Equation 4 defines how the transverse position $\mathbf{y}$ of the ray changes along the ray. Thus $(\mathbf{y} - A\mathbf{y}_O - \mathbf{e})/B$ is the *positional* invariant of the ray. Similarly Equation 5 defines how the reduced inclination $\boldsymbol{\alpha}$ changes along the ray; $(\boldsymbol{\alpha} - C\mathbf{y}_O - \boldsymbol{\pi})/D$ is the *inclinational* invariant of the ray.

For the situation represented by Figure 1(a) the positional invariant is listed in Table 1 and the inclinational invariant in Table 2.

By a similar argument one can determine the two invariants for each of the other three situations represented in Figure 1. They are also listed in Tables 1 and 2.

Figure 1(b) represents a ray through system S from a distant object point O. The entrance plane of S is now not T_O but T_0. All the rays from O have the same reduced inclination $\boldsymbol{\alpha}_O$. Choosing one of the rays is now equivalent to choosing the transverse position $\mathbf{y}_0$ of the ray at T_0.

In Figure 1(c) there is an image point I with transverse position $\mathbf{y}_I$ on the exit plane T_I of system S while in (d) I is distant and rays to it all have reduced inclination $\boldsymbol{\alpha}_I$. For each of these cases the positional and inclinational invariants are listed in Tables 1 and 2 respectively.

In what follows we make use only of the positional invariants. Reference to inclinational invariants has been introduced largely for mathematical completeness. We note, however, that the right-hand sides of Equations 4 and 5 can be equated. This gives a relationship between reduced inclination $\boldsymbol{\alpha}$ and transverse position $\mathbf{y}$ of a ray at any longitudinal position which can be solved for one in terms of the other. The same can be done for all the systems represented in Figure 1.

We turn now to the relationship of the transverse position of a ray in one transverse plane to its transverse position in another.

Table 1 Positional invariant along a ray through an object (O) or image (I) point and scalar magnification $X_{\mathbf{y},1\to2}$ and transverse translation $\mathbf{d}_{\mathbf{y},1\to2}$ of transverse position $\mathbf{y}$ from transverse plane T_1 to transverse plane T_2.

Point	Positional invariant	Scalar magnification $X_{\mathbf{y},1\to2}$	Transverse translation $\mathbf{d}_{\mathbf{y},1\to2}$
O on entrance plane	$\boldsymbol{\alpha}_O = (\mathbf{y} - A\mathbf{y}_O - \mathbf{e})/B$	B_2/B_1	$(A_2 - X_{\mathbf{y},1\to2}A_1)\mathbf{y}_O + \mathbf{e}_2 - X_{\mathbf{y},1\to2}\mathbf{e}_1$
O distant	$\mathbf{y}_0 = (\mathbf{y} - B\boldsymbol{\alpha}_O - \mathbf{e})/A$	A_2/A_1	$(B_2 - X_{\mathbf{y},1\to2}B_1)\boldsymbol{\alpha}_O + \mathbf{e}_2 - X_{\mathbf{y},1\to2}\mathbf{e}_1$
I on exit plane	$\boldsymbol{\alpha}_I = (-\mathbf{y} + D(\mathbf{y}_I - \mathbf{e}))/B + \boldsymbol{\pi}$	B_2/B_1	$(D_2 - X_{\mathbf{y},1\to2}D_1)\mathbf{y}_I - D_2\mathbf{e}_2 + X_{\mathbf{y},1\to2}D_1\mathbf{e}_1 + B_2(\boldsymbol{\pi}_2 - \boldsymbol{\pi}_1)$
I distant	$\mathbf{y}_0 = (\mathbf{y} + B(\boldsymbol{\alpha}_I - \boldsymbol{\pi}))/D + \mathbf{e}$	D_2/D_1	$-(B_2 - X_{\mathbf{y},1\to2}B_1)\boldsymbol{\alpha}_I - D_2(\mathbf{e}_2 - \mathbf{e}_1) + B_2\boldsymbol{\pi}_2 - X_{\mathbf{y},1\to2}B_1\boldsymbol{\pi}_1$

Table 2 Inclinational invariant along a ray through an object (O) or image (I) point and scalar magnification $X_{\alpha,1\to2}$ and transverse displacement $\mathbf{d}_{\alpha,1\to2}$ of reduced inclination $\boldsymbol{\alpha}$ from transverse plane T_1 to transverse plane T_2.

Point	Inclinational invariant	Scalar magnification $X_{\boldsymbol{\alpha},1\to2}$	Transverse translation $\mathbf{d}_{\boldsymbol{\alpha},1\to2}$
O on entrance plane	$\boldsymbol{\alpha}_O = (\boldsymbol{\alpha} - C\mathbf{y}_O - \boldsymbol{\pi})/D$	D_2/D_1	$(C_2 - X_{\boldsymbol{\alpha},1\to2}C_1)\mathbf{y}_O + \boldsymbol{\pi}_2 - X_{\boldsymbol{\alpha},1\to2}\boldsymbol{\pi}_1$
O distant	$\mathbf{y}_0 = (\boldsymbol{\alpha} - D\boldsymbol{\alpha}_O - \boldsymbol{\pi})/C$	C_2/C_1	$(D_2 - X_{\boldsymbol{\alpha},1\to2}D_1)\boldsymbol{\alpha}_O + \boldsymbol{\pi}_2 - X_{\boldsymbol{\alpha},1\to2}\boldsymbol{\pi}_1$
I on exit plane	$\boldsymbol{\alpha}_I = (\boldsymbol{\alpha} + C(\mathbf{y}_I - \mathbf{e}))/A + \boldsymbol{\pi}$	A_2/A_1	$-(C_2 - X_{\boldsymbol{\alpha},1\to2}C_1)\mathbf{y}_I + C_2\mathbf{e}_2 - X_{\boldsymbol{\alpha},1\to2}C_1\mathbf{e}_1 - A_2(\boldsymbol{\pi}_2 - \boldsymbol{\pi}_1)$
I distant	$\mathbf{y}_0 = (-\boldsymbol{\alpha} + A(\boldsymbol{\alpha}_I - \boldsymbol{\pi}))/C + \mathbf{e}$	C_2/C_1	$(A_2 - X_{\boldsymbol{\alpha},1\to2}A_1)\boldsymbol{\alpha}_I + C_2(\mathbf{e}_2 - \mathbf{e}_1) - A_2\boldsymbol{\pi}_2 + X_{\boldsymbol{\alpha},1\to2}A_1\boldsymbol{\pi}_1$

Positional transformation from one transverse plane to another

Consider Figure 1(a) again. Let us re-label transverse plane T as T_1. The system from T_O to T_1 is now system S_1. System S_1 has transference $\mathbf{T}_1$, scalar fundamental properties A_1, B_1, C_1 and D_1 and 2×1 fundamental properties $\mathbf{e}_1$ and $\boldsymbol{\pi}_1$. Adding subscript 1 to all the variables on the right-hand side of Equation 4 we obtain the positional invariant at transverse plane T_1.

We can also choose a second transverse plane T_2 in Figure 1(a) in which everything in the previous paragraph holds if we replace subscript 1 by 2. The positional invariants at T_1 and T_2 can be equated because they are both equal to the incident reduced inclination $\boldsymbol{\alpha}_O$:

$$\left(\mathbf{y}_1 - A_1\mathbf{y}_O - \mathbf{e}_1\right)/B_1 = \left(\mathbf{y}_2 - A_2\mathbf{y}_O - \mathbf{e}_2\right)/B_2 . \qquad (6)$$

Solving for the emergent position we obtain

$$\mathbf{y}_2 = \frac{B_2}{B_1}\mathbf{y}_1 + \left(A_2 - \frac{B_2}{B_1}A_1\right)\mathbf{y}_O + \mathbf{e}_2 - \frac{B_2}{B_1}\mathbf{e}_1 \qquad (7)$$

which we can write

$$\mathbf{y}_2 = X_{\mathbf{y},1\to2}\mathbf{y}_1 + \mathbf{d}_{\mathbf{y},1\to2} \qquad (8)$$

where

$$X_{\mathbf{y},1\to2} = B_2/B_1 \qquad (9)$$

and

$$\mathbf{d}_{\mathbf{y},1\to2} = \left(A_2 - X_{\mathbf{y},1\to2}A_1\right)\mathbf{y}_O + \mathbf{e}_2 - X_{\mathbf{y},1\to2}\mathbf{e}_1 . \qquad (10)$$

Equations 8 to 10 are Equations 11 to 13 of the previous paper[1] with astigmatism ignored. The important difference is that $\mathbf{X}_{\mathbf{y},1\to2}$ has been replaced by $X_{\mathbf{y},1\to2}$. The linear component[8] of the affine magnification[9] associated with the referral now reduces to the much more familiar scalar magnification.

Equation 8 tells us that, from transverse plane T_1 to transverse plane T_2, the transverse position $\mathbf{y}_1$ of a ray gets multiplied by the scalar $X_{\mathbf{y},1\to2}$ and has a fixed vector $\mathbf{d}_{\mathbf{y},1\to2}$ added to it. In other words if we follow a ray from one transverse plane to another we find that its transverse position relative to the longitudinal axis Z undergoes magnification by the factor $X_{\mathbf{y},1\to2}$ followed by transverse shift $\mathbf{d}_{\mathbf{y},1\to2}$. We call $X_{\mathbf{y},1\to2}$ the *positional magnification* and $\mathbf{d}_{\mathbf{y},1\to2}$ the *common transverse translation* from T_1 to T_2 although we shall often use abbreviated terminology. The positional magnification (Equation 9) and the common transverse translation (Equation 10) are listed in Table 1 for the case of object O on the entrance plane.

Just as there is transformation of position there is also transformation of reduced inclination of a ray from one transverse position to another which we now consider briefly in passing.

Inclinational transformation from one transverse plane to another

Starting with Equation 5 instead of Equation 4 and applying a similar argument one finds that the reduced inclination changes from transverse plane T_1 to transverse plane T_2 in a similar manner. It turns out that

$$\boldsymbol{\alpha}_2 = X_{\boldsymbol{\alpha},1\to2}\boldsymbol{\alpha}_1 + \mathbf{d}_{\boldsymbol{\alpha},1\to2} . \qquad (11)$$

$\boldsymbol{\alpha}_1$ is magnified by scalar $X_{\boldsymbol{\alpha},1\to2}$ and has a fixed vector $\mathbf{d}_{\boldsymbol{\alpha},1\to2}$ added to it. Expressions for inclinational magnification $X_{\boldsymbol{\alpha},1\to2}$ and fixed shift $\mathbf{d}_{\boldsymbol{\alpha},1\to2}$ for O on the entrance plane are listed in Table 2.

So far we have examined only the case represented by Figure 1(a), namely for referral by an object point O on the entrance plane. The other three situations represented by Figure 1 are handled in a similar fashion. Equations 8 and 11 are obtained again for all three cases but with expressions for $X_{\mathbf{y},1\to2}$ and $\mathbf{d}_{\mathbf{y},1\to2}$ as given in Table 1 and with expressions for $X_{\boldsymbol{\alpha},1\to2}$ and $\mathbf{d}_{\boldsymbol{\alpha},1\to2}$ as given in Table 2.

Apertures may be of any shape. We turn now to apertures that are elliptical (including circular) in particular.

Elliptical apertures

Figure 2 shows an ellipse E in a transverse plane T. a and b are half the major and minor diameters of E and $\mathbf{v}_a$ and $\mathbf{v}_b$ are unit vectors along those diameters. For the present we assume $a > b > 0$. The centre of E has position vector $\mathbf{y}^0$ in T relative to the longitudinal axis Z. A point is shown on E with position vector $\mathbf{y}$ relative to Z and position vector $\mathbf{r}$ relative to the centre. Then

$$\mathbf{r} = \mathbf{y} - \mathbf{y}^0 \quad (12)$$

is a radial vector of the ellipse. Ellipse E has equation

$$\mathbf{r}^{\mathrm{T}}\mathbf{M}\mathbf{r} = 1 \quad (13)$$

where $\mathbf{M}$ is positive definite, that is, both its eigenvalues are greater than zero. (This is the same as Equation 19 of the previous paper[1].) $\mathbf{M}$ represents the shape, size and orientation of E; we refer to it simply as the *geometry* of E. $\mathbf{M}^{1/2}$ is the principal square root of $\mathbf{M}$ and

$$\mathbf{R}_{\mathrm{p}} = \mathbf{M}^{-1/2} \quad (14)$$

the positive definite generalized radius of E. Equation 13 can be written

$$\left(\mathbf{R}_{\mathrm{p}}^{-1}\mathbf{r}\right)^{\mathrm{T}} \mathbf{R}_{\mathrm{p}}^{-1}\mathbf{r} = 1\,. \quad (15)$$

$\mathbf{R}_{\mathrm{p}}$ can be obtained from the spectral theorem[10] of linear algebra:

$$\mathrm{R}_{\mathrm{p}} = a\mathbf{v}_a\mathbf{v}_a^{\mathrm{T}} + b\mathbf{v}_b\mathbf{v}_b^{\mathrm{T}}\,. \quad (16)$$

An example is treated in the Appendix of the previous paper[1].

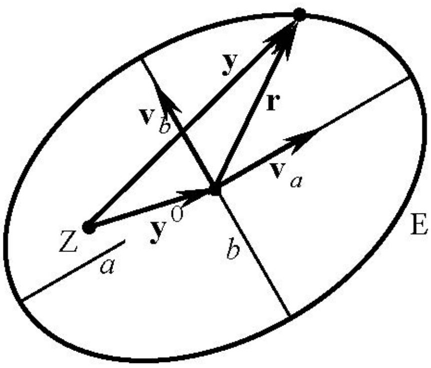


Figure 2 An ellipse E in a transverse plane T. a is half the major diameter and b half the minor diameter. $\mathbf{v}_a$ and $\mathbf{v}_b$ are unit vectors along the major and minor diameters respectively. Z is the longitudinal axis and is orthogonal to the plane of the paper. Relative to Z the centre of the ellipse has position vector $\mathbf{y}^0$. $\mathbf{y}$ is the position vector of a point on the ellipse. $\mathbf{r} = \mathbf{y} - \mathbf{y}^0$ is the position vector of the point relative to the centre of the ellipse; it is a radial vector of the ellipse.

Suppose in transverse plane T there is a thin opaque plate with an elliptical aperture A with centre $\mathbf{y}^0$ and positive definite generalized radius $\mathbf{R}_{\mathrm{p}}$. We define A to be all the points in T for which

$$\mathbf{r}^{\mathrm{T}}\mathbf{M}\mathbf{r} \le 1\,. \quad (17)$$

Equality implies points on the margin.

An aperture may be referred to another transverse plane by an object or image point. The question is: What is the geometry and location of the referred aperture?

Aperture referral

Consider now an elliptical aperture A_1 in transverse plane T_1. Its centre is at $\mathbf{y}_1^0$, its geometry is $\mathbf{M}_1$ and its positive definite generalized radius is $\mathbf{R}_{\mathrm{p}1}$. A radial vector $\mathbf{r}_1$ has equation

$$\mathbf{r}_1^{\mathrm{T}}\mathbf{M}_1\mathbf{r}_1 = 1 \quad (18)$$

or

$$\left(\mathbf{R}_{\mathrm{p}1}^{-1}\mathbf{r}_1\right)^{\mathrm{T}} \mathbf{R}_{\mathrm{p}1}^{-1}\mathbf{r}_1 = 1\,. \quad (19)$$

Suppose aperture A_1 is referred to another transverse plane T_2 by an object or image point to become referred aperture A_2. Every position vector $\mathbf{y}_1$ in $\mathbf{A}_1$ undergoes affine magnification to $\mathbf{y}_2$ in $\mathbf{A}_2$ according to Equation 8. In particular $\mathbf{y}_1^0$ becomes

$$\mathbf{y}_2^0 = X_{\mathbf{y},1\to2}\mathbf{y}_1^0 + \mathbf{d}_{\mathbf{y},1\to2}\,. \quad (20)$$

Applying Equations 8 and 20 to Equation 12 we see that radial vector $\mathbf{r}_1$ of $\mathbf{A}_1$ undergoes scalar magnification according to

$$\mathbf{r}_2 = X_{\mathbf{y},1\to2}\mathbf{r}_1\,. \quad (21)$$

Solving for $\mathbf{r}_1$ and substituting into Equation 18 we obtain

$$\mathbf{r}_2^{\mathrm{T}}\mathbf{M}_2\mathbf{r}_2 = 1 \quad (22)$$

where

$$\mathbf{M}_2 = \frac{\mathbf{M}_1}{X_{\mathbf{y},1\to2}^2}\,. \quad (23)$$

Equation 22 shows that referred aperture A_2 is also elliptical; its centre is at $\mathbf{y}_2^0$ given by Equation 20 and its geometry is given by Equation 23. In fact Equation 21 shows that referral magnifies aperture A_1 by the scalar $X_{\mathbf{y},1\to2}$; thus referral changes the size but not the shape or orientation of the ellipse.

Substituting for $\mathbf{r}_1$ in Equation 19 we find that

$$\left(\mathbf{R}_2^{-1}\mathbf{r}_2\right)^{\mathrm{T}} \mathbf{R}_2^{-1}\mathbf{r}_2 = 1 \quad (24)$$

where

$$\mathbf{R}_2 = X_{\mathbf{y},1\to2}\mathbf{R}_{\mathrm{p}1} \quad (25)$$

is a generalized radius of A_2. The positive definite generalized radius is

$$\mathbf{R}_{\mathrm{p}2} = \sqrt{X_{\mathbf{y},1\to2}^2}\,\mathbf{R}_{\mathrm{p}1}\,. \quad (26)$$

Circles are simpler to handle than ellipses and the pupil of the eye and most other apertures of relevance in optometry are close to circular. Accordingly we now examine that special case.

Circular apertures

If $a = b > 0$ then ellipse E is a circle of radius a. We represent the radius by r_p, that is, $r_p = a$. Subscript p (for positive) distinguishes this as the radius of a circle as conventionally defined. We refer to it as the positive radius of the circle in contrast to other radii (without subscript p) which may be positive or negative. Ellipse E has geometry

$$\mathbf{M} = \mathbf{I} / r_p^2 \tag{27}$$

and positive definite generalized radius

$$\mathbf{R}_p = \mathbf{I} r_p . \tag{28}$$

Its equation simplifies to

$$\mathbf{r}^T \mathbf{r} = r_p^2 . \tag{29}$$

A circular aperture A_1 with positive radius r_{p1} and centre at $\mathbf{y}_1^0$ in transverse plane T_1 becomes referred aperture A_2 in T_2. A_2 is circular with centre given by Equation 20 and radius

$$r_2 = X_{\mathbf{y},1\to2} r_{p1} \tag{30}$$

which is positive, zero or negative according as the magnification $X_{\mathbf{y},1\to2}$ is positive, zero or negative. The positive radius of A_2 is

$$r_{p2} = \sqrt{X_{\mathbf{y},1\to2}^2}\, r_{p1} . \tag{31}$$

The following simple example provides insight into the significance of negative magnification of an aperture in particular.

Example

Figure 3 represents an example of referral of a circular aperture. The longitudinal axis Z is the optical axis of the system. Circular aperture A_1 of positive radius r_{p1} is referred to transverse plane T_2 by an object point O on Z. A_2 is the referred aperture. T_2 is a distance z downstream from A_1 and the intervening medium has index n. The transference of system S_1 from object plane T_O to the plane of A_1 can be written

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{I}A_1 & \mathbf{I}B_1 \\ \mathbf{I}C_1 & \mathbf{I}D_1 \end{pmatrix} \tag{32}$$

where we have deleted the fifth row and column (which we can do because both $\mathbf{e}$ and $\boldsymbol{\pi}$ are null). The transference of system S_2 from T_O to T_2 is

$$\mathbf{T}_2 = \begin{pmatrix} \mathbf{I}(A_1 + \zeta C_1) & \mathbf{I}(B_1 + \zeta D_1) \\ \mathbf{I}C_1 & \mathbf{I}D_1 \end{pmatrix} \tag{33}$$

a result obtained in the usual way. Here $\zeta = z / n$.

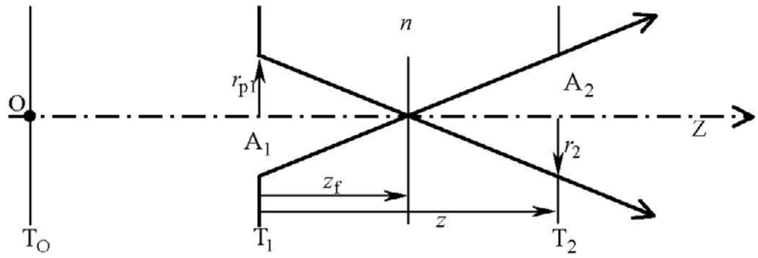


Figure 3 Object point O on longitudinal axis Z refers circular aperture A_1 of radius $r_{p1} = r_1 > 0$ in transverse plane T_1 to transverse plane T_2 at distance z downstream in a homogeneous medium. Referred aperture A_2 is circular with radius r_2. As z increases r_2 decreases reaching 0 for $z = z_f$ and then goes negative. For $z = 2z_f$ (as in the figure) $r_2 = -r_1$. The positive radius of A_2 is $r_{p2} = \sqrt{r_2^2}$.

From Equation 9 we obtain the scalar magnification

$$X_{\mathbf{y},1\to2} = 1 + \zeta D_1 / B_1 . \tag{34}$$

Also, from Equation 10,

$$\mathbf{d}_{\mathbf{y},1\to2} = \mathbf{o} \tag{35}$$

because $\mathbf{y}_O = \mathbf{e}_2 = \mathbf{e}_1 = \mathbf{o}$. As z increases from zero referred aperture A_2 initially decreases in size in Figure 3; this implies that system S_1 has D_1 and B_1 of opposite signs. From Equations 30 and 34 we see that referred aperture A_2 has radius

$$r_2 = (1 + \zeta D_1 / B_1) r_{p1} . \tag{36}$$

Solving we find that r_2 reduces to zero when $\zeta = \zeta_f$ where

$$\zeta_f = -B_1 / D_1 . \tag{37}$$

Equation 36 can then be written

$$r_2 = (1 - \zeta / \zeta_f) r_{p1} . \tag{38}$$

It is clear that r_2 is positive, zero or negative according as ζ is less than, equal to or greater than ζ_f. A_2 has positive radius

$$r_{p2} = \sqrt{(1 - \zeta / \zeta_f)^2}\, r_{p1} . \tag{39}$$

Note that r_2 decreases linearly with z; the function is smooth. On the other hand r_{p2} decreases linearly until $z = z_f$ where the derivative is not defined after which it increases linearly.

Figure 3 is drawn for $\zeta = 2\zeta_f$. In that case the magnification is $X_{y,1\to2} = -1$ (from Equations 34 and 37) and $r_2 = -r_{p1}$ (from Equation 38) and $r_{p1} = r_{p2}$ (from Equation 39). Figure 4 shows the view along longitudinal axis Z. Points P_1, Q_1 and R_1 on the margin of A_1 become points P_2, Q_2 and R_2, respectively, on the margin of A_2; referral preserves the anticlockwise order of the points. In effect referral causes rotation through $180°$. However if points are ignored referred aperture A_2 is identical to aperture A_1.

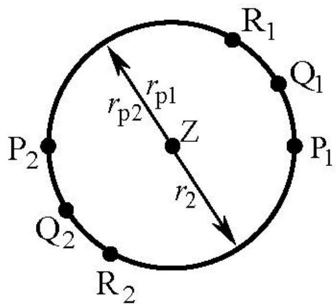


Figure 4 View along longitudinal axis Z of aperture A_1 and referred aperture A_2 in Figure 3. A_1 is circular with positive radius r_{p1}. The scalar magnification associated with the referral is $X_{y,1\to2} = -1$ and, hence, A_2 has (negative) radius $r_2 = -r_{p1}$ and positive radius $r_{p2} = r_{p1}$. Points P_1, Q_1 and R_1 in anticlockwise order on the margin of A_1 become points P_2, Q_2 and R_2, respectively, also in anticlockwise order on the margin of A_2. In effect referral causes rotation through $180°$.

Concluding remarks

We have here specialized the results of an earlier paper[1] by ignoring astigmatism. Referral from transverse plane T_1 to transverse plane T_2 results in displacement of a point with position vector $\mathbf{y}_1$ in T_1 to a point with position vector $\mathbf{y}_2$ given by Equation 8. The transformation is a special case of affine magnification[9]; there is scalar magnification by $X_{y,1\to2}$ followed by transverse translation $\mathbf{d}_{y,1\to2}$ expressions for which are given for the four situations in Table 1. Thus eliminating astigmatism simplifies the linear magnification[8] $\mathbf{X}_{y,1\to2}$ of the general case[1] to the conceptually much simpler and more familiar scalar magnification $X_{y,1\to2}$. 2×2 matrices have been eliminated in Equation 8 and in Table 1 (and Table 2); only scalars and vectors remain. The mathematics becomes more accessible for persons less familiar with matrix algebra.

Scalar magnification $X_{y,1\to2}$ may be positive, zero or negative. If it is positive referral magnifies the aperture in the familiar sense; there is change in size but not in shape or orientation. If $X_{y,1\to2}$ is zero the referred aperture is a point. If $X_{y,1\to2}$ is negative there is inversion through its centre and magnification by $-X_{y,1\to2}$. Inversion effectively rotates the shape through $180°$ about Z.[8] If such rotation represents a symmetry of the aperture then referral does not change the shape or orientation. Figure 4 illustrates the case of $X_{y,1\to2} = -1$ and a circular aperture; the aperture and the referred aperture are identical. The same would be true of an elliptical aperture and an aperture of any other shape with Z an axis of two-fold rotational symmetry. However the same is not true of apertures of other shapes as illustrated in Figure 5; triangular aperture $P_1Q_1R_1$ in transverse plane T_1 undergoes referral and magnification $X_{y,1\to2} = -1$ to become referred aperture $P_2Q_2R_2$ in transverse plane T_2. In contrast to the case in Figure 4 the referred aperture is distinguishable from the aperture. Note that the effect is to turn the shape through $180°$. To say that the shape is turned upside down would be misleading.

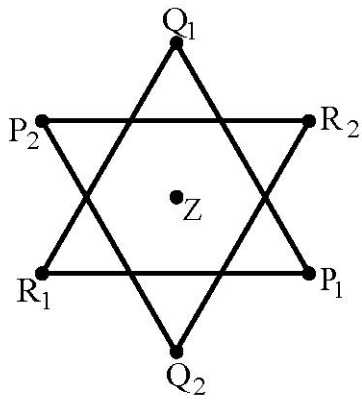


Figure 5 Triangular aperture $P_1Q_1R_1$ in transverse plane T_1 of Figure 3 and referred aperture $P_2Q_2R_2$. The positional magnification is $X_{y,1\to2} = -1$. Referral rotates the shape through $180°$. In contrast to the circular aperture in Figure 4 the referred aperture is distinguishable from the aperture. To describe the change as turning the shape upside down is not correct for that would require R_2 and P_2 to be interchanged.

Consider an eye with stigmatic elements. The retinal blur patch for a distant object point is the pupil referred to the retina by the object point. Associated with the referral is scalar magnification that is positive, zero or negative according as the eye is myopic, emmetropic or hyperopic.

Acknowledgements

The author acknowledges support from the National Research Foundation and continuing discussions with T Evans, RD van Gool and A Rubin.

References

1. Harris WF. Aperture referral in heterocentric astigmatic systems. *Ophthal Physiol Opt* 2011 **31** 603-614.
2. Harris WF. Symplecticity and relationships among the fundamental properties in linear optics. *S Afr Optom* 2010 **69** 3-13.
3. Harris WF. Effective corneal patch of an astigmatic heterocentric eye. *Ophthal Physiol Opt* 2011 **31** 79-90.
4. Harris WF. Optical axes of eyes and other optical systems. *Optom Vis Sci* 2009 **86** 537-541.
5. Wallace DAR. *Groups, Rings and Fields*. Springer-Verlag, London, 1998.
6. Gallian JA. *Contemporary Abstract Algebra*. 3rd ed, Heath, Lexington, MA, 1994.
7. Herstein, IN. *Abstract Algebra*. 3rd ed, Wiley, New York, 1999.
8. Harris W F. Generalized magnification in visual optics. Part 1: Magnification as linear transformation. *S Afr Optom* 2010 **69** 109-122.
9. Harris W F. Generalized magnification in visual optics. Part 2: Magnification as affine transformation. *S Afr Optom* 2010 **69** 166-172.
10. Laub AJ. *Matrix Analysis for Scientists and Engineers*. Society for Industrial and Applied Mathematics, Philadelphia, 2005 97.