Yves Le Grand on matrices in optics with application to vision: Translation and critical analysis

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Abstract

An appendix to Le Grand’s 1945 book, Optique Physiologique: Tome Premier: La Dioptrique de l’Œil et Sa Correction, briefly dealt with the application of matrices in optics. However the appendix was omitted from the well-known English translation, Physiological Optics, which appeared in 1980. Consequently the material is all but forgotten. This is unfortunate in view of the importance of the dioptric power matrix and the ray transference which entered the optometric literature many years later. Motivated by the perception that there has not been enough care in optometry to attribute concepts appropriately this paper attempts a careful analysis of Le Grand’s thinking as reflected in his appendix. A translation into English is provided in the appendix to this paper. The paper opens with a summary of the basics of Gaussian and linear optics sufficient for the interpretation of Le Grand’s appendix which follows. The paper looks more particularly at what Le Grand says in relation to the transference and the dioptric power matrix though many other issues are also touched on including the conditions under which distant objects will map to clear images on the retina and, more particularly, to clear images that are undistorted. Detailed annotations of Le Grand’s translated appendix are provided. (S Afr Optom 2013 72(4) 145-166)

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Introduction

Matrix methods have played significant roles in modern ophthalmic optics. In particular the dioptric power matrix was introduced into optometry by Long in 1976 and the system matrix or ray transference by Keating in 1981. As pointed out by Blendowske, however, much of the material had actually been published before by Fick in the early 1970s in a series of 22 short articles; one in particular describes the dioptric power matrix. Because he ‘published in German and additionally in a journal more related to the craftsman than to a scientific readership [Fick’s] approach was nearly forgotten’. Indeed there is earlier material still which unfortunately seems to have has suffered the same fate, also, apparently, for not having been published in English: an appendix entitled ‘Le Calcul des Matrices en Optique’ (pages 322-328) to the book Optique Physiologique: Tome Premier: la Dioptrique de l’Œil et Sa Correction published by Le Grand in 1945. The 3rd edition of the book (dated 1964), with
minor changes to the appendix, appeared in English translation\textsuperscript{28} in 1980 as the well-known \textit{Physiological Optics} by Le Grand and El Hage\textsuperscript{29}. Sadly, however, the appendix was omitted from the English version.

Le Grand’s appendix is interesting in its own right. However it is not easy to come by, is terse, is in French, gives few explanations and uses symbolism that differs from that in current use. For these reasons it seems appropriate to make it available in English translation for the modern vision scientist. Our purpose here, then, is to do so and to provide a critical analysis of the appendix set in the context of matrix methods in current use in optometry and vision science.

We begin with the basics of Gaussian and linear optics expressed in the matrix symbolism in recent usage. With that as reference we then examine Le Grand’s appendix. His appendix is given in English translation in the appendix to this paper together with detailed annotations.

Elements of matrix methods in Gaussian and linear optics

Gaussian and linear optics are both first-order optical models. The first is effectively a two-dimensional optics (Figure 1). Rays can be visualized in a single plane, the plane containing the optical axis and the object and image points; usually it is a representative plane in systems invariant under rotation about the optical axis. The second model is a three-dimensional generalization. Only the latter is adequately capable of handling astigmatism, a fundamentally three-dimensional phenomenon. For an excellent account of both models of optics, and their relationship to geometrical and other models, the reader is referred elsewhere\textsuperscript{30}. We summarize here basic results which we shall need in our analysis of Le Grand’s appendix below.

The transference

The matrix of interest goes by several names including system matrix\textsuperscript{22, 23, 31-33}, matrix of the system\textsuperscript{34, 35}, lens matrix\textsuperscript{36}, \textit{ABCD} matrix\textsuperscript{37}, optical matrix\textsuperscript{37}, ray-transfer matrix\textsuperscript{37, 38} and (ray) transference\textsuperscript{15-18}. We shall refer to it here as the \textit{transference} and represent it by the symbol \( S \). It is a complete representation of the first-order optical character of a system\textsuperscript{37}.

In Gaussian optics the transference is a real \( 2 \times 2 \) matrix which we write as\textsuperscript{30, 38}

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

(1)

where \( A, B, C \) and \( D \) are what we call the four fundamental (first-order optical) properties of Gaussian system \( S \).\textsuperscript{39, 41} \( A \) is the \textit{dilation}, \( B \) the \textit{disjugacy}, \( C \) the \textit{divergence} and \( D \) the \textit{divarication}. The four fundamental properties are not independent but are related by the equation

\[
AD - CB = 1.
\]

(2)

In other words the transference has a unit determinant\textsuperscript{30, 38}. We say that the \( 2 \times 2 \) transference has three degrees of freedom. If we know three of the fundamental properties we can usually calculate the remaining one. We cannot always do so, however; for example we cannot calculate \( D \) if we know \( A, B \) and \( C \) and \( A \) happens to be 0.

In linear optics the transference expands to become a \( 4 \times 4 \) real matrix which is usually written\textsuperscript{9, 37}

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]

(3)

in terms of four \( 2 \times 2 \) submatrices \( A, B, C \) and \( D \), the four fundamental properties of the linear system. Other properties can be obtained from the fundamental properties; they are \textit{derived} properties. For example the dioptric power of a system is defined by\textsuperscript{39}

\[
F = -C.
\]

(4)

The \( 4 \times 4 \) transference has 10 degrees of freedom, the 16 entries being related by six scalar equations instead of just one (Equation 2). The six equations are contained within the single matrix equation\textsuperscript{30, 37, 42-48}

\[
S^T E S = E
\]

(5)

where \( S^T \) represents the matrix transpose of \( S \),

\[
E = \begin{pmatrix}
O & I \\
-I & O
\end{pmatrix}
\]

(6)

and \( O \) and \( I \) are \( 2 \times 2 \) null and identity matrices respectively. Substitution from Equations 3 and 6 into Equation 5 leads to three matrix equations,

\[
A^T C = C^T A,
\]

(7)

\[
B^T D = D^T B,
\]

(8)
A^T D - C^T B = I. \tag{9}

If we multiply these equations out in terms of their entries we find that Equations 7 and 8 are each equivalent to a single scalar equation and Equation 9 is equivalent to four scalar equations.

The $2 \times 2$ transference (Equation 1) also obeys Equation 5 if $O$ and $I$ are interpreted as $1 \times 1$ null and identity matrices, that is, simply as the scalars 0 and 1. It also obeys Equations 7 and 8 but trivially and Equations 2 and 9 are identical.

Any $2n \times 2n$ matrix $S$ which obeys Equation 5, where $O$ and $I$ are $n \times n$, is said to be symplectic\textsuperscript{42-46}. In particular both $2 \times 2$ and $4 \times 4$ transferences are symplectic. Many results for transferences arising as a consequence of their symplecticity are summarized elsewhere\textsuperscript{47}; some will be used below.

Like Le Grand\textsuperscript{26,27} we shall be concerned here with optical systems all of whose component refracting elements are centred on a common axis, the optical axis. For systems with centred elements one can make use of uncentered\textsuperscript{49} or augmented symplectic matrices\textsuperscript{50-53} which have an extra row and an extra column.

A ray traversing a system

Figure 1 represents a ray traversing a Gaussian system $S$. $S$ lies between entrance and exit planes $T_0$ and $T$. The elements of $S$ (none of which is shown) are centred on optical axis $Z$. Upstream of $S$ the medium has index of refraction $n_0$; downstream from $S$ the index is $n$. Transverse positions and inclinations are measured relative to $Z$. At incidence the ray has state\textsuperscript{30,37}

$$\rho_0 = \begin{pmatrix} y_0 \\ a_0 \end{pmatrix}, \tag{10}$$

a $2 \times 2$ matrix, where $y_0$ is the transverse position,

$$a_0 = n_0 a_0 \tag{11}$$

and $a_0$ is the inclination. We call $a_0$ the reduced inclination of the ray at incidence. At emergence the ray has state $\rho$ defined similarly. The transference $S$ of $S$ is an operator that changes $\rho_0$ across $S$ to $\rho$

according to\textsuperscript{30,37}

$$Sp_0 = \rho, \tag{12}$$

which is equivalent to the pair of scalar equations

$$Ay_0 + B a_0 = y \tag{13}$$

$$Cy_0 + D a_0 = a. \tag{14}$$

Figure 1 An arbitrary Gaussian optical system $S$ lies between entrance plane $T_0$ and exit plane $T$. None of its refracting elements is shown. All elements are centred on the optical axis $Z$. A ray enters $S$ with transverse position $y_0$ and inclination $a_0$ and emerges with transverse position $y$ and inclination $a$. The ray is confined to the plane of the paper; $Z$ lies in the same plane. The media before and after the system have indices of refraction $n_0$ and $n$.

We now generalize these equations for linear optics; surfaces need not be invariant under rotation about the optical axis, that is, they may be astigmatic. Transverse position $y_0$ becomes a vector\textsuperscript{30,37}

$$y_0 = \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} \tag{15}$$

with Cartesian coordinates $y_{10}$ and $y_{20}$ which we usually take as horizontal and vertical components respectively, and similarly for $y$, $a_0$, $a$, $a_0$, and $a$. $y_0$ and $y$ are the transverse positions, $a_0$ and $a$ the inclinations and $a_0$ and $a$ the reduced inclinations of the ray at incidence and emergence. The components of $a_0$ and $a$ are also called direction cosines\textsuperscript{37} and the components of $a_0$ and $a$ are also known as optical direction cosines\textsuperscript{37}. The incident state of the ray (Equation 10) generalizes to the $4 \times 1$ partitioned matrix

$$\rho_0 = \begin{pmatrix} y_0 \\ a_0 \end{pmatrix} \tag{16}$$

and similarly for the emergent state $\rho$. Equation 12 retains the same form. Scalar Equations 13 and 14 become the matrix equations

$$Ay_0 + B a_0 = y \tag{17}$$

$$Cy_0 + D a_0 = a \tag{18}$$

Linear optics includes Gaussian optics as a special case. In the absence of astigmatism the fundamental properties become scalar matrices; we can write
\[ A = AI \]  

\[ \text{for example.} \]

**Systems in series**

Consider two optical systems \( S_1 \) and \( S_2 \) with transferences \( S_1 \) and \( S_2 \) respectively. Suppose they are juxtaposed to form a compound system. Light traverses \( S_1 \) first and then \( S_2 \). It follows from Equation 12 that the compound system has transference

\[ S = S_2S_1. \]  

(20)

In general the transference of a compound system is the product of the transferences of component systems in reverse order.

Two elementary systems are a homogenous gap and a single refracting surface. In Gaussian optics their transferences are

\[ \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix} \]  

(21)

and

\[ \begin{pmatrix} 1 & 0 \\ C & 1 \end{pmatrix} \]  

(22)

respectively. \( \zeta = z/n \) is the *reduced* width of the gap, \( z \) the actual width and \( n \) the index of refraction.

\( C = -F \) is the divergence and \( F \) the dioptric power of the surface.

**Object and image**

Suppose an object point \( O \) maps to an image point \( I \) through a Gaussian system \( S \) (Figure 2). Relative to entrance plane \( T_0 \) the longitudinal position of \( O \) is \( z_O \). Consider the compound system \( S_D \) from the object plane \( T_0 \) to the image plane \( T_1 \). It consists of a homogeneous gap of width \( -z_O \) (the minus sign because \( z_O \) is negative), system \( S \) (with transference \( S \)) and a homogeneous gap of width \( z_I \). It follows from Equation 20 that the compound system has transference

\[ S_D = \begin{pmatrix} 1 & \zeta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 1 & -\zeta_O \\ 0 & 1 \end{pmatrix} \]  

(23)

where \( \zeta_O = z_O/n_0 \) and \( \zeta_1 = z_I/n \). Multiplying out we obtain

\[ S_D = \begin{pmatrix} A + \zeta_1C & B + \zeta_1D - (A + \zeta_1C)\zeta_O \\ C & D - C\zeta_O \end{pmatrix}. \]  

(24)

We now apply Equation 13 but across compound system \( S_D \). Because every ray from \( O \) arrives at the same point \( I \) it follows that the disjugacy \( B_D \) of \( S_D \) must be zero. \( S_D \) is a *conjugate* system. Hence, from Equation 24,

\[ B + \zeta_1D - (A + \zeta_1C)\zeta_O = 0 \]  

(25)

from which it follows that

\[ \zeta_1 = \frac{A\zeta_O - B}{-C\zeta_O + D}. \]  

(26)

**Astigmatism**

In linear optics the transferences of the homogeneous gap and refracting surface are

\[ \begin{pmatrix} I & \zeta \end{pmatrix} \]  

(27)

and

\[ \begin{pmatrix} I & O \\ C & I \end{pmatrix} \]  

(28)

respectively. Here \( C = -F \) where \( F \) is the (symmetric) dioptric power matrix of the possibly-astigmatic surface. Using the notation introduced elsewhere we can write the power of the surface in principal meridional form as \( F_1 \{\theta\}F_2 \{\theta+90^\circ\} \). (We read this as ‘\( F_1 \) along \( \theta \) and \( F_2 \) along \( \theta+90^\circ \).’ We sometimes abbreviate this to \( F_1 \{\theta\}F_2 \). The dioptric power matrix is given by

\[ F = \begin{pmatrix} f_{11} & f_{12} \\ f_{11} & f_{12} \end{pmatrix} \]  

(29)

where

\[ f_{11} = f_{21} = (F_1 - F_2)\sin\theta\cos\theta \]  

(30)

\[ f_{12} = f_{21} = (F_1 - F_2)\sin^2\theta \]  

(31)

\[ \text{Figure 2 An object point O forms an image point I through a Gaussian system S.} \]
$$f_{21} = F_1 \sin^2 \theta + F_2 \cos^2 \theta.$$  (32)

Equivalent equations are given by others. As for Gaussian optics the transferences of systems compounded of gaps and surfaces can be obtained by multiplying elementary transferences (Equations 27 and 28) in reverse order.

**Thick spectacle lens**

A thick possibly-bitoric spectacle lens in front of the eye constitutes an instrument $S_C$ of four elements: in order they are a refracting surface of power $F_1$, a homogeneous gap (the body of the lens) of reduced width $\zeta_2$, a second refracting surface of power $F_3$ and a second homogeneous gap (between spectacle lens and eye) of reduced width $\zeta_4$. The transference is

$$S_C = \begin{pmatrix} I & \zeta_4 I & I & O \\ O & I & C_3 & I \\ I & O & I & C_1 \\ C_1 & I & I & I \end{pmatrix},$$  (33)

that is,

$$S_C = (I + \zeta_2 C_1 + \zeta_4 (C_3 I + \zeta_2 C_1) + C_1) \begin{pmatrix} I & \zeta_4 I \\ C_3 (I + \zeta_2 C_1) + C_1 & C_3 \zeta_2 + I \end{pmatrix}.$$  (34)

**Condition for sharp retinal images**

Consider an emmetropic retinal eye in linear optics. Its transference $S$ is given by Equation 3. Rays from a distant object point $O$ enter the eye with different transverse positions $y_0$ and yet arrive at the same point (transverse position $y$) on the retina it follows from Equation 17 that

$$A = O.$$  (35)

Suppose a spectacle lens or other device compensates for the refractive error of an ametropic eye. Then the compound system of device and eye satisfies Equation 35. Equation 35 is the necessary and sufficient condition for sharp retinal images. Furthermore Equation 9 results in

$$B = -C^{-T}$$  (36)

and, hence, Equation 17 becomes

$$-C^{-T} \alpha_0 = y.$$  (37)

A system for which Equation 35 holds we call exit-plane focal.

**Condition for sharp undistorted images**

Equation 37 is a linear mapping from $\alpha_\theta$ to $y$. It shows that, although the retinal image is sharp, it may be distorted. In general a distant circle maps to an ellipse on the retina. It maps to a circle if and only if $C^{-T}$ is a scalar multiple of $R$ where $R$ is either a rotation matrix

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$  (38)

or a reflection matrix

$$\overline{R}_\theta = \begin{pmatrix} \sin 2\theta & \cos 2\theta \\ \cos 2\theta & -\sin 2\theta \end{pmatrix}.$$  (39)

Because $R^{-T}$ is itself a rotation or reflection matrix we can say that the object circle maps to an image circle on the retina if and only if $C$ or $F$ is a scalar multiple of $R$, that is

$$C = mR.$$  (40)

for a scalar $m$. Thus, in addition to Equation 35, there are two conditions for sharp undistorted images on the retina: either (a) the diagonal elements of $C$ are equal and the off-diagonal elements are equal in magnitude but opposite in sign (based on Equation 38) or (b) the off-diagonal elements are equal and the diagonal elements are equal in magnitude but opposite in sign (Equation 39).

Suppose there is an observer beyond the plane of the distant circle who traces the object circle in a clockwise sense. The image circle on the retina would be traced in the same sense if condition (a) were satisfied and in the opposite sense if condition (b) were satisfied. We express this by saying that condition (a) preserves chirality and condition (b) reverses chirality.

For the naked eye the divergence $C$ (or power $F$) is usually not very different from a scalar matrix. It is quite possible for an eye to satisfy condition (a) but it seems inconceivable for an eye to satisfy condition (b). For eyes, then, it seems safe to disregard condition (b) and conclude that condition (a) is the only condition in practice for sharp undistorted retinal images. One expects the same to be true of the eye compensated by
a contact or spectacle lens. (In Footnote [82] a thick lens is described which satisfies condition (b) but is totally unrealistic.) However it is quite possible for condition (b) to be satisfied by the compound system of eye and more complicated optical instrument.

**Condition for sharp, undistorted and unrotated images**

An eye satisfying Equation 35 and condition (a) is emmetropic and retinal images of distant objects are undistorted. However they are rotated through angle \( \theta \). For the image to be sharp, undistorted and unrotated it must be that \( \theta = 0 \) in which case \( R_\theta = I \); in other words \( C \) and \( F \) must be scalar matrices. Usually they are close to scalar matrices and, so, rotations are likely to be small.

**Inverse transference**

Because of its symplecticity the transference is always invertible, the inverse being\(^{41, 47}\)

\[
S^{-1} = \begin{pmatrix}
D^T & -B^T \\
-C^T & A^T
\end{pmatrix}.
\]  

(41)

From Equation 20 we have

\[
S^{-1} = S_I^{-1}S_2^{-1}
\]  

(42)

which shows that for a compound system the order of multiplication of inverse transferences is the same as order of component systems.

As an operator the inverse transference gives the incident state of a ray in terms of its emergent state:

\[
S^{-1}p = p_0.
\]  

(43)

**Condition for compensation of refractive error by means of a thin lens**

Consider the compound system of eye and thin lens in front of it. If the eye has transference given by Equation 3 and the lens has divergence \( C_1 \) (a symmetric matrix) and is located at reduced distance \( \varsigma_1 \) in front of the eye then the compound system has transference

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
I & I_{\varsigma_1} \\
O & I
\end{pmatrix}
\begin{pmatrix}
I & O \\
C_1 & I
\end{pmatrix}
\]

Making use of Equation 47 we can write Equation 49 as

\[
F_0A_C + C_C = O.
\]  

(51)

From Equation 49 we obtain

\[
C_C = -B^{-1}A_A.
\]  

(52)

Hence Equation 50 becomes

\[
(C - DB^{-1}A)A_C = mR.
\]  

(53)

We recognize the coefficient of \( A_C \) as the Schur
complement47, 55, 56 of C in the symplectic matrix S. Hence47, 57 Equation 53 simplifies to

\[-B^{-T}A_C = mR\]  \hspace{1cm} (54)

and so

\[A_C = -mB^TR.\]  \hspace{1cm} (55)

We substitute into Equation 49 and use the fact that \(AB^T\) is symmetric to obtain

\[C_C = mA^TR.\]  \hspace{1cm} (56)

**Le Grand’s appendix**

The brief summary above of the matrix approach to Gaussian and linear optics provides a coherent framework against which we can now examine Le Grand’s appendix. A translation is given in the appendix of this paper. It is a translation of the appendix in the first edition of Le Grand’s book26. A few minor changes made in the third edition27 are noted. Superscript numbers, as in \(n\), in the translation continue to refer to references listed under References at the end of the paper; superscript numbers in square brackets, as in \([n]\), refer to detailed annotations listed as footnotes in the translation. Le Grand’s original equation numbering has been preserved including the fact that his equation numbers appear to the left of equations; the numbers were changed in the third edition.

We turn now to the more important questions concerning Le Grand’s appendix. Many minor matters, including a small number obvious typographical errors, will be left to the footnotes.

**Le Grand on the transference**

The first reference Le Grand makes to what appears to be a transference is his unnumbered equation after superscript [26]. Comparison of the matrix on right-hand side with Equation 1 suggests that the matrix is a transference in Gaussian optics. This is reinforced by Le Grand’s Equation (108) in the light of the symplectic requirement, Equation 2. More particularly, comparison of the left-hand matrix with Equation 22 suggests that it is the transference of a refracting surface. However the signs of the bottom-left entries differ.

At first sight the second matrix at superscript [30] appears to be the transference of a homogeneous gap of reduced thickness \(-\delta\) (compare Equation 21) and the equation appears to represent the transference of a compound system consisting of the homogeneous gap followed by a system \(S\) obtained by multiplication in reverse order according to Equation 20. Similar remarks apply to the equations at [32] and [34].

In actual fact the negative sign gives Le Grand’s matrix away, not as the transference \(S\) of a gap of reduced thickness \(-\delta\) but the inverse transference \(S^{-1}\) of a gap of thickness \(\delta\). Similarly Le Grand’s matrix for the refracting surface is actually the inverse transference of the surface.

While transferences multiply in reverse order of component elements (Equation 20) inverse transferences multiply in the same order as the component elements (Equation 42). This accounts for the order of multiplication in Equation (110) at superscript [34], in the numerical example that follows it and elsewhere in the appendix.

In the presence of astigmatism Le Grand writes the \(4\times4\) matrix \(M\) (at [63]) with the six equations among the entries (at [67]). That it is also an inverse transference rather than a transference is clear from the matrix for a gap (at [70]) and the order of multiplication at [71]. It is also clear at [69] where the bottom-left block of the matrix for an astigmatic refracting surface consists of powers and not their negatives. The inverse of a symplectic matrix is also symplectic43 which is why there should be the six equations Le Grand refers to as mentioned above (just after Equation 4) for a transference.

The inverse transference is an operator that operates on the emergent state of a ray to give the incident state for that ray for a given optical system (Equation 43). However nowhere in his appendix does Le Grand hint at such an interpretation. One gets the impression that he sees the matrix merely as a convenient mathematical device for determining optical properties of compound systems. The matrix is that of course, but its nature as an operator which represents the way the system operates on light seems to be an important feature of the inverse transference, as of the transference itself, which does not come across here.

Instead of thinking of the transference as the property of a particular system that operates on a ray
traversing it Le Grand’s thinking is in terms of object and image. This is clear from his frequent reference to origins (see just before [25] and [27] for example). Origins relative to which objects and images are located need to be defined. There is less emphasis on the need to define optical systems. An optical system followed by a homogeneous gap is treated between [27] and [30] but it is not treated as a new compound system; it is treated as the same system but with the origin for images shifted downstream by the width of the gap. Of course the two approaches are equivalent but Le Grand’s seems less clear.

An advantage of Le Grand’s approach is the simpler equations one can obtain if one chooses the origins to be at the principal planes. He takes advantage of this fact at [17] and in the solution to Exercise 43. However this is true only for Gaussian optics; because principal planes are not usually well defined in the presence of astigmatism the approach does not generalize to linear optics.

**Le Grand on other matters**

At [72] Le Grand presents the condition that an eye or compensated eye will form sharp retinal images of distant objects. The condition consists of four equations. An explanation is not given and is certainly not obvious. Le Grand’s conditions are derived in Footnote [72]; they are a direct consequence of the condition (Equation 35) given above.

In the presence of astigmatism the sharp retinal image is usually distorted. At [75] Le Grand presents a supplementary condition, also without explanation, which needs to be satisfied if the image is undistorted. His condition takes the form of a single scalar equation. The required conditions are derived in Footnote [75]; they take the form of three scalar equations in addition to Le Grand’s single equation. The basis of the derivation being Equation 40 and the two associated conditions (a) and (b). One obtains a pair of equations (Equations 49 and 50) which can be reduced to Equations 55 and 56. There is much more to the problem than presented by Le Grand; strictly speaking his condition is neither necessary nor sufficient. It is possible to have systems which produce sharp undistorted images of distant objects but which do not satisfy Le Grand’s condition. However it seems unlikely for such cases to be encountered in conventional applications in optometry. If these cases are neglected Le Grand’s condition becomes necessary. Strictly speaking it is not a sufficient condition although one expects the additional requirement to be satisfied automatically by most eyes.

It is perhaps worth noting that the numerical example (at [83] to [85]) which Le Grand selects to illustrate the use of his equations at [82] can be solved simply by means of Gaussian optics applied once along each principal meridional plane. That is because there are no obliquely-crossing principal meridians. However his equations are actually more powerful than he illustrates; they can handle obliquely-crossing meridians without any difficulty which is not possible with Gaussian optics.

**Concluding remarks**

Authors in optometry have not always been as careful as they might have been in attributing credit.
The recognition that the present author is no exception has motivated this study. The attempt has been to make a scholarly analysis of exactly what Le Grand had to say in his appendix about matrices in optics and, more particularly, about the dioptric power matrix and the ray transference. In order to obviate any possible ambiguity I have thought it appropriate to provide detailed and exhaustive annotations to the translation.

In view of what we have seen here, if asked whether Le Grand described the dioptric power matrix and the transference one would have to say ‘No, not in the full sense in which those concepts are now understood’. That, however, would be a misrepresentation for he certainly came very close. The missing \( \frac{1}{2} \) in his expression for the off-diagonal elements of the dioptric power matrix of an astigmatic refracting surface is almost certainly a typographical error and is not the issue. He seems to have thought of three separate powers rather than an integrated whole dioptric power. Le Grand worked not with the transference but its inverse; nowhere in his appendix does he deal with the inverse of his matrix (which would have been the transference) and, importantly, he does not seem to have thought of his matrix as a whole entity in its own right and more particularly as an entity that operates on the state of a ray traversing the system.

When reading Le Grand it is important to note that he takes the optical system of the eye to be from just anterior to the cornea to just posterior to the lens. The vitreous is excluded. (See Footnote [36].) The eye as optical system to which we refer in this paper includes the vitreous.

The conditions for sharp retinal images of distant objects are derived in Footnote [72]; they are identical to the conditions that Le Grand presents without justification. The conditions for sharp images that are also undistorted are derived in Footnote [75]; here Le Grand’s condition does not tell the whole story. Nevertheless it seems safe to say his condition tells most of the story at least in the context of conventional optometric applications. A fuller analysis of sharp and undistorted retinal images, including for near objects, has recently appeared elsewhere.

In looking carefully at Le Grand’s thinking, as we have tried to do here, we become more conscious still of those scholars before him who too have not always been given their due. There are Herzberger’s important contributions of the 1930s, Smith’s of the 1920s and many others all the way back to Gauss and, perhaps, before. For, reading these older works, one is frequently surprised, by ideas one thought were new. “The wind goeth toward the south, and turneth about unto the north; it whirleth about continually, and the wind returneth again according to his circuits. / … / The thing that hath been, it is that which shall be; and that which is done is that which shall be done: and there is no new thing under the sun. / Is there anything whereof it may be said, See, this is new? It hath been already of old time, which was before us.” (Ecclesiastes 1 9-10).

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I thank an anonymous reviewer for an insightful analysis and for drawing my attention to the existence of a Spanish translation of Le Grand’s appendix. Acquisition of a copy, and the assistance of a Spanish speaker familiar with ophthalmic optics, may give me more insight into Le Grand’s work.
Appendix


In preparing this volume I have strived to keep the mathematical development at a level as elementary as possible, and in terms of optical formulae most widely used in classical education. However, the same problems can be solved with greater elegance and generality by employing certain algebraic symbols, matrices, the use of which is current in contemporary physics. We will consider here only square matrices.

One calls a matrix of order \( n \) a table formed of \( n \) rows and \( n \) columns, consisting thus of \( n^2 \) quantities arranged as follows:[3]

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{pmatrix}
\]

In the term \( a_{pq} \) of the matrix index \( p \) designates the number of the row and index \( q \) the number of the column. For what follows the only property of matrices which we will need concerns multiplication: one defines the product of two matrices of order \( n \) as a third matrix also of order \( n \)

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\
b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn}
\end{pmatrix}
\]

the terms of the matrix product obtained by the \( n^2 \) relations[4]

\[c_{pq} = \sum_{k=1}^{n} a_{pk} b_{kq};\]

tag that is to say that any term \( c \) is obtained from the \( a \)s of the same row and the \( b \)s of the same column as the sum of their pairwise products.

It is easy to see that the product of matrices is not commutative; one cannot reverse the factors. For example

\[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\begin{pmatrix}
5 & 6 \\
7 & 8
\end{pmatrix} = \begin{pmatrix}
19 & 22 \\
43 & 50
\end{pmatrix},
\]

while

\[
\begin{pmatrix}
5 & 6 \\
7 & 8
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix} = \begin{pmatrix}
23 & 34 \\
31 & 46
\end{pmatrix}.
\]

However this operation is associative: in the product of several matrices one can group them at will provided one does not change the order:

\[
(M_1 M_2) M_3 = M_1 (M_2 M_3).\]

Gauss’s Coefficients. – Consider a centred[6] system on the axis of which we choose any two origins; the abscissas \( x \) and \( x' \) of two conjugate points

---

[1]. This is a translation by WFH from the original French of *Annexe II: Le Calcul des Matrices en Optique*, pp 322-328, and parts of pp 332 & 341 of the 1st edn of the book in which LG (Le Grand) originally wrote ‘in 1945 as a textbook for the students of the Institute of Optics in Paris’. The 3rd edn of the book appeared in English translation in 1980. Although *Annexe II* appears with minor changes on pp 364-381 of the 3rd edn, it is omitted in the English version.

[2]. The title page of the 1st edn is headed, in translation, ‘Yves LE GRAND/ Deputy Director of the Physical Laboratory of the National Museum of Natural History/ Head of Conferences at the Polytechnic School’. In the 3rd edn LG is described as ‘Professor of the National Museum of Natural History/ Examiner of students of the Polytechnic School’. From the French version of Wikipedia (27 March 2013) we learn that LG (1908-1986) obtained his doctorate in 1936 with a thesis on dispersion of light in the eye. The same year he joined the Laboratory of Physics Applied to Living Beings of the National Museum of Natural History of which he later became Director. From 1942 he taught physiological optics at the School of Optics. In 1966 he was elected president of the French Society of Physiological Optics which he had helped to create.

[3]. Throughout LG uses straight lines for matrices (as now often used for determinants) instead of the curved brackets used here. There is a typographical error in the 3rd edn: the 2nd and 3rd entries \( a_{n2} \) and \( a_{n3} \) of the bottom row of this matrix are each given as \( a_{n1} \).

[4]. The last subscript \( q \) in this equation is incorrectly written \( p \) in the 3rd edn.

[5]. LG uses multiplication signs (\( \times \)) and \( M \) is represented by \( \mathbf{M} \).

[6]. By a centred system LG means what is often called a symmetric system. Refracting elements of the system are invariant under rotation about a common axis, the optical axis.
projected onto the axis[7] are linked by a homographic transformation[8] which we will write in the form:[9]

\[ x' = \frac{cx + d}{ax + b} \]  \hspace{1cm} (107)

The quantities[10] \( a, b, c, d \) are evidently determined only to a constant factor since the relation still holds when one multiplies them by the same number. One will therefore be able to impose an arbitrary relation among these quantities.

The longitudinal magnification is expressed by:

\[ \frac{dx'}{dx} = \frac{b - d}{(ax + b)^2} \]  \hspace{1cm} (108)

If the extreme media have an index of refraction of 1 the transverse magnification \( y'/y \) is such that[11]

\[ \left( \frac{y'}{y} \right)^2 = \frac{dx'}{dx} \]  \hspace{1cm} (109)

consequently, if we choose the condition

\[ bc - ad = 1 \]  \hspace{1cm} (108)[12]

as arbitrary relation, then it becomes (one proves that it necessarily takes the sign +)[13]

\[ \frac{y'}{y} = \frac{1}{ax + b} \]  \hspace{1cm} (109)[14]

Quantities \( a, b, c, d \), are called Gauss's coefficients; \( a \) is a power, \( b \) and \( c \) numbers, and \( d \) a length[15]. From a knowledge of these coefficients, one immediately deduces the cardinal elements of the system[16]; the absissas of the principal points are obtained by putting \( y'/y = 1 \) in equation (109), which gives \( x_0 = (1 - b)/a \) and by substitution in (107) \( x'_0 = (c - 1)/a \). If one takes these points as origins,

[7]. The picture is as follows:

Object point O maps to an image point I through system S.

[8]. Also called a Möbius transformation.

[9]. The equation number is given as (AII-1) in the 3rd edn. Apparently this equation was better known in the past than it is today. A derivation was given by Pendlebury[70]. We derive it above (see Eq 26).

[10]. After Eq (108) is applied these quantities become entries of the transference (Eq 1) of system S. In particular \( a = -C \), \( b = D \), \( c = A \), \( d = -B \).

[11]. The 3rd edn has 'see exercise 3' in brackets. The compound system from object plane to image plane has transference

\[ \begin{pmatrix} A + Cx' & 0 \\ C & D - Cx \end{pmatrix} \]  \hspace{1cm} (Eqs 24 & 25). Application of Eq 13 to the compound system shows that \( (A + Cx')y = y' \). Because of symplecticity (Eq 2)

2) \( (A + Cx')(D - Cx) = 1 \). Hence, the transverse magnification is \( y'/y = A + Cx' = 1/(D - Cx) \) in agreement with Eq (109).

[12]. The equation number is (AII-2) in the 3rd edn. Because of Eq (108) the matrix \( \begin{pmatrix} b & d \\ a & c \end{pmatrix} \) to be encountered below has unit determinant and is symplectic.

[13]. The plus sign follows the closing bracket in the original. This is corrected in the 3rd edn.

[14]. Eq (AII-3) in the 3rd edn.

[15]. Although \( d \) has the dimension length it does not seem to be entirely accurate to call \( d \) a length. For an air gap it is the negative of the length of the gap. In general it is the negative of the disjugacy. What LG calls Gauss’s coefficients are the entries of the inverse of the transference (see Footnote 26).

[16]. Because of the assumption of unit refractive index outside the system (see 2nd sentence of this paragraph) the nodal points coincide with the principal points expressions for the locations of which are given here. Setting \( x = 0 \) in Eq (107) locates the emergent focal point at \( x' = d / b \).

The incident focal point is obtained by setting \( x' = 0 \) in Eq (107), the result being \( x = -d / c \).
it happens that \( b = c = 1, \ d = 0 \) and equation (107) becomes\(^{[17]}\)

\[
\frac{1}{x'} = \frac{1}{x} + a
\]

which shows us that \( a \) is nothing other than the true power\(^{[18]}\) of the system. If one is satisfied with the choice of conjugate origins then \( d = 0 \), and from (108) \( bc = 1 \).\(^{[19]}\) Then equation (107) is written\(^{[20]}\)

\[
\frac{1}{x'} = \frac{b^2}{x} + ba;
\]

we recover the frontal\(^{[21]}\) notations, \( b \) designating the frontal factor\(^{[22]}\) \( g' \) and \( ba \) the frontal power \( D'_f \).\(^{[23]}\)

If the indices of refraction of the extreme media differ from 1 the expressions above remain valid\(^{[24]}\) provided that \( x \) and \( x' \) designate the reduced distances of the points considered from the respective origins.

**Use of matrices.** – The arithmetic of matrices allows the very simple determination of Gauss’s coefficients of a centred system, whatever its complexity.

1. If the system is a refracting surface of power \( D \), the origins coinciding with the apex of the surface,\(^{[25]}\) one has

\[
\frac{1}{x'} = \frac{1}{x} + D; \quad \text{consequently} \quad a = D, \quad b = c = 1, \quad d = 0, \quad \text{which can be written}\(^{[26]}\)

\[
\begin{pmatrix}
1 & 0 \\
D & 1
\end{pmatrix}
= \begin{pmatrix}
b & d \\
0 & 1
\end{pmatrix}
\]


---

\([17]\). The system under consideration has now changed but the symbolism remains the same. Originally the system was defined by the dashed vertical lines in the following sketch; it is now defined by the incident \( P \) and the emergent \( P' \) principal planes whose longitudinal positions \( x_0 \) and \( x'_0 \) are now 0. It follows from \( x_0 = (1-b)/a \) and \( x'_0 = (c-1)/a \) that \( b = c = 1 \). \( d = 0 \) because the new system is conjugate.

\[\text{O}\]
\[\text{P}\]
\[\text{S}\]
\[\text{I}\]
\[\text{P'}\]
\[\text{x}\]
\[\text{x'}\]

---

\([18]\). ‘la puissance vraie’ which is changed to ‘la puissance équivalente’ (equivalent power) in the 3rd edn.

\([19]\). Again the system changes but the symbols remain the same. \( P \) and \( P' \) in the sketch above are not necessarily principal planes but any pair \( Q \) and \( Q' \) of conjugate planes. \( d = 0 \) because this new system, from \( Q \) to \( Q' \), is also conjugate and, hence, \( bc = 1 \) follows. LG (p 38 of Ref 26) called \( 1/x \) proximité. LG&EH (Le Grand and El Hage) (p 13 of Ref 29) used proximity and ascribed the concept and term to Herschel in 1827. They also comment (p 14) that proximity and power are sometimes called vergence and convergence respectively, terms best avoided because of the other meanings they have in binocular vision. Presumably the reference is to John Frederick William Herschel (1792-1871, lived in South Africa 1834-1838), the son of Frederick William Herschel who discovered Uranus and who died in 1822.

\([20]\). The equation follows directly from Eq (107) with \( d = 0 \) and \( bc = 1 \). It is Eq 48 (except that \( D'_f \) is \( D' \) there), together with Eq 52, on p 105 of Ref 26 and Eq 1.50 on p 22, together with Eq 1.49 on p 21, of Ref 29.

\([21]\). LG uses frontal instead of vertex.

\([22]\). ‘le facteur frontale’ which becomes ‘le facteur de forme’ (form factor) in the 3rd edn.

\([23]\). \( D'_f \) is the back (or image) frontal power (p 102 of Ref 26 and p 20 of Ref 29) usually known as back-vertex power. LG (pp 102-105 of Ref 26 and pp 19-22 of Ref 29) defines 4 frontal powers: forward frontal object power (front vertex power), forward frontal image power, rear frontal object power and rear frontal image power (back vertex power) but retains only the 1st and last. Footnote 2 on p 104 of Ref 26 mentions that \( b \) was called effectivity factor by Smith and shape factor by Ogle. In Ref 63 Smith refers to ‘so-called’ “effectivity” factors—not a very suitable name’. For Ogle see Ref 71.

\([24]\). ‘valables’ which is changed to ‘exactes’ (correct) in the 3rd edn.

\([25]\). We have interpreted the French un dioptre de puissance \( D \) as meaning a refracting surface of power \( D \). It appears that \( x \) and \( x' \) here are reduced distances. If the indices of refraction either side of the surface were 1 then the power of the surface would be 0. Dioptre meaning refracting surface is distinct from LG’s dioptre which means dioptre (diopter in American English) the unit of power. \( D \) is the symbol LG uses for power (puissance).

\([26]\). Using the values from Footnote [10] we obtain \( \begin{pmatrix} b & d \\ a & c \end{pmatrix} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} \). Comparing this with Eq 1 we see that LG’s matrix is not the transference \( S \) but its inverse \( S^{-1} \). LG’s matrix \( \begin{pmatrix} 1 & 0 \\ D & 1 \end{pmatrix} \) applies to a thin system (a refracting surface or thin lens) and to any Gaussian system but taken between its principal planes.
2. Let us displace the origin of the images by reduced distance \( \delta \);[27] the system[28], which before had coefficients \( a_1, b_1, c_1, d_1 \), has \( a, b, c, d \) such that[29]

\[
x' = x_1 - \delta = \frac{c_1 x + d_1}{a_1 x + b_1} - \delta = \frac{(c_1 - a_1 \delta) x + (d_1 - b_1 \delta)}{a_1 x + b_1}
\]

\[
= \frac{cx + d}{ax + b},
\]

from which

\[
a = a_1, \ b = b_1, \ c = c_1 - a_1 \delta, \ d = d_1 - b_1 \delta \quad (\text{one can verify that condition (108) is satisfied}), \text{which can be written}^{[30]}
\]

\[
\begin{pmatrix}
  b_1 & d_1 \\
  a_1 & c_1
\end{pmatrix}
\begin{pmatrix}
  1 & -\delta \\
  0 & 1
\end{pmatrix}
= \begin{pmatrix}
  b & d \\
  a & c
\end{pmatrix}.
\]

3. Let us link a second system \( a_2 b_2 c_2 d_2 \) (with origins \( O_2 O'_2 \)) to the first system \( a_1 b_1 c_1 d_1 \) (with origins \( O_1 O'_1 \)) and let us consider the resulting system \( abcd \) (with origins \( OO' \)); let us make \( O \) coincide with \( O_1 \) and \( O' \) with \( O'_2 \). We have

\[
x' = \frac{cx + d}{ax + b} = \frac{c_2 x_1' + d_2}{a_2 x_1' + b_2}, \quad \text{with} \quad x_1' = \frac{c_1 x + d_1}{a_1 x + b_1}^{[31]}
\]

from which by identification

\[
\begin{align*}
a &= a_1 b_2 + c_1 a_2 \\
b &= b_1 b_2 + d_1 a_2 \\
c &= a_1 d_2 + c_1 c_2 \\
d &= b_1 d_2 + d_1 c_2,
\end{align*}
\]

which can be written[32]

\[
\begin{pmatrix}
  b_1 & d_1 \\
  a_1 & c_1
\end{pmatrix}
\begin{pmatrix}
  b_2 & d_2 \\
  a_2 & c_2
\end{pmatrix}
= \begin{pmatrix}
  b & d \\
  a & c
\end{pmatrix}.
\]

4. If we add to the three preceding results the associative character of matrix multiplication we obtain the following rule: consider a centred system made up of refracting surfaces of powers \( D_1, D_3, D_5, \ldots \) separated by reduced distances \( \delta_2, \delta_4, \ldots \); the resulting system, of which the object origin coincides with the apex of the first refracting surface and the image origin with the apex of the last, has Gauss’s coefficients[33]

\[
(110)^{[34]} \quad \begin{pmatrix}
  1 & 0 & 1 & -\delta_2 \\
  D_1 & 1 & 0 & 1 \\
  0 & 1 & -\delta_4 & 1 \\
  D_3 & 1 & 0 & 1 \\
  \vdots & \vdots & \vdots & \vdots
\end{pmatrix}
= \begin{pmatrix}
  b & d \\
  a & c
\end{pmatrix}.
\]

Applications. – Expression (110) allows the immediate recovery of the formulae associated with Gullstrand’s[1],[35] One can also use it for the numerical calculation

---

[27] The origin for \( x' \) is shifted a longitudinal distance \( n' \delta \) as in the sketch below. \( n' \) is the index of refraction of the medium after the optical system.

[28] What was optical system \( S \) now becomes system \( S_1 \). System \( S \) now includes system \( S_1 \) and the homogeneous gap of width \( n' \delta \).

[29] \( x, x', x'_1 \) are reduced distances.

[30] For the reader familiar with transferences the order of multiplication here appears incorrect. If the gap in the sketch has transference \( S_2 \) then the transference of system \( S \) is \( S = S_2 S_1 \) (Eq 20). Because LG is multiplying inverse transferences (see Footnote [26]), his order is correct. See Eq 42.

[31] The prime is missing from \( x'_1 = (c_1 x + d_1)/(a_1 x + b_1) \) in the 3rd edn.

[32] See Footnote [30].

[33] See Footnote [30].

[34] (All-4) in the 3rd edn.

[35] ‘(’ refers to the footnote on the same page (p 325) which reads ‘(’). See exercises 43 and 44. ’ The exercises follow the Anmexe on p 329.
of a complex system; for example let us apply it to the case of the theoretical non-accommodated eye:\[36\]

\[D_1 = 48.3462 \text{ dt}\]
\[\delta_2 = 0.55 \times 10^{-3} / 1.3771 = 3.9939 \times 10^{-4} \text{ m}\]
\[D_3 = -6.1077 \text{ dt}\]
\[\delta_4 = 3.05 \times 10^{-3} / 1.3374 = 2.2805 \times 10^{-3} \text{ m}\]
\[D_5 = 8.0980 \text{ dt}\]
\[\delta_6 = 4 \times 10^{-3} / 1.42 = 2.8169 \times 10^{-3} \text{ m}\]
\[D_7 = 14 \text{ dt}\]

One calculates the successive products step by step; thus the products of the first three, the first five, and of the seven matrices equal\[37\]

\[
\begin{pmatrix}
1.00244 & -3.9939 \times 10^{-4} \\
42.3564 & 0.98069 \\
0.98069 & -2.6855 \times 10^{-3} \\
49.5158 & 0.88410
\end{pmatrix},
\]

respectively. The position of the object principal point will be

\[
\frac{1 - b}{a} = \frac{1 - 0.90442}{59.9404} = 1.5946 \times 10^{-3} \text{ m};\[38\]
\]

one will find the image principal point\[39\] in the same way: we obtain the values in table I (page 50)\[40\].

Moreover there will be every interest in preserving the origins of Gauss’s coefficients;\[41\] the object origin will be the apex of the cornea, a well-defined physical\[42\] point, and the image origin will be the apex of the posterior surface of the crystalline lens, which moves very little during accommodation. If the eye is corrected one will introduce the correcting refracting surfaces in front of the eye and formulae (110) and (109) will allow us very easily to calculate the dimension of the retinal image,\[43\] assumed

\[36\]. Note that this ‘eye’ excludes the vitreous body. The index of refraction and width of the vitreous are not given in the *Annexe*. If we take the index to be 1.336 (as given in tableau 1 on p 50 of Ref 26) and the width as 24.1965 mm (the value calculated on p 49 of Ref 26 to make the eye emmetropic) then we find that the transference of the whole eye, including the vitreous, is

\[
\begin{pmatrix}
0.00000 & 16.6832 \times 10^{-3} \text{ m} \\
-59.9404 \text{ D} & 0.90442
\end{pmatrix}
\]

That the top left entry is 0 shows that the eye is indeed emmetropic. The inverse of the eye’s transference is

\[
\begin{pmatrix}
0.90442 & -16.6832 \times 10^{-3} \text{ m} \\
59.9404 \text{ D} & 0.00000
\end{pmatrix}
\]

also that dioptries is abbreviated *dt* and not *D*. LG uses the decimal comma rather than the decimal point. Here and in some places elsewhere he uses : instead of / for division. Fewer significant digits are given in the 3rd edn; corresponding to the numbers here are the numbers 48.35, 0.55, 1.3771, 3.99, 6.11, 3.05, 1.3374, 2.28, 8.10, 4.10, 1.42, 2.82, 14.

\[37\]. In Ref 26 the 1st 4 is missing from the bottom left entry of the 1st matrix and the last digit of the bottom right entry of the 3rd matrix is 2 instead of 1. The 4 and the 1 are made bold here. In the 3rd edn the matrices are given as

\[
\begin{pmatrix}
1.002 & -3.994 \times 10^{-4} \\
2.356 & -0.9807 \\
49.52 & 0.88410
\end{pmatrix},
\]

respectively, with the leading 4 again missing in the bottom-left entry of the 1st matrix and minus signs inserted in the bottom-right entries of the 1st and 2nd matrices.

\[38\]. *m* for metres was missing in the original. This is the distance posterior from the 1st surface of the eye. In the 3rd edn the position is given as 1.59 \times 10^{-3}.

\[39\]. The original has *le p p image*. The image (or emergent) principal point has reduced position (\(e - t\))/\(a = (0.74461 - 1)/95.9404 = -4.2607 \times 10^{-3} \text{ m} = 4.3224 \text{ mm} \) anterior to the 2nd surface of the lens of the eye. The index of refraction the vitreous is not given in the appendix. *Tableau 1* on p 50 of Ref 26 LG gives it as 1.336. Using this value we find that the actual position is 5.6922 mm anterior to the 2nd surface of the crystalline lens of eye. The same table positions the latter surface 7.6 mm from the 1st surface of the eye. Hence the emergent principal point is 1.9078 mm into the eye in agreement with the number given in tableau 1.

\[40\]. In the original, tableau 1 extends across pp 50 & 51 of Ref 26. The relevant data appear in the column with heading *Théorique* and subheading *non accom*. The power and the locations of the principal points are given in the set of rows of the table marked as *Œil complet* on p 51. Reference to the page number is omitted in the 3rd edn; the table is on pp 74 & 75 of that edn and many of the numbers there have fewer significant digits.

\[41\]. Gauss’s coefficients depend on the system and, hence, on the location of the system’s entrance and exit planes. The origins are really for longitudinal positions \(x\) and \(x'\) rather than for the coefficients themselves.

\[42\]. The original has *concret*.

\[43\]. One applies Eq (110), starting with the 1st surface of the compensating lens in front of the eye and ending with the posterior surface of the lens of the eye. The resulting matrix gives \(a\) and \(b\) for use in Eq (109). With the longitudinal position \(x\) of the object relative to the 1st surface of the eye and the height \(y\) of the object Eq (109) gives the height \(y'\) of the image.
perfect\textsuperscript{[44]}; in the case of a remote object the condition of correction is simply written according to (107):\textsuperscript{[45]}
\begin{equation}
\frac{c}{a} = x',
\end{equation}
\textsuperscript{(111)}\textsuperscript{[46]}
x' designating the reduced distance from the posterior pole of the crystalline lens to the retina. The use of matrices will greatly simplify the writing of ophthalmic calculations. For example K. N. Ogle\textsuperscript{[47]} used them successfully for studying aniseikonia (1936)\textsuperscript{[48]}.

\textit{Astigmatic systems.} – Let us now consider a system consisting of astigmatic refracting surfaces\textsuperscript{[49]} possessing a common normal (which one takes as axis Ox) and such that every plane section of one of these refracting surfaces with a plane passing through Ox is symmetric with respect to this axis\textsuperscript{[50]}, let us choose any axes Oy\textsubscript{1} and Oz\textsubscript{1} forming a trirectangular trihedron with Ox\textsuperscript{[51]}.

We know that one of these refracting surfaces is defined by its principal powers \(D_y\) and \(D_z\) and by the angle \(\phi\) which the principal section \(yO\) makes with plane \(xOy_1\).\textsuperscript{[52]} Instead of these disparate quantities

\[\frac{c}{a} = x',\]

\[\frac{c}{a} = x',\]

\[\frac{c}{a} = x',\]

\[\frac{c}{a} = x',\]

\[\frac{c}{a} = x',\]

\[\frac{c}{a} = x',\]

\[\frac{c}{a} = x',\]

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we will introduce three powers\[^{[53]}\]

\[
A = (D_y - D_z) \sin 2\varphi
\]

\[
B = D_y \sin^2 \varphi + D_z \cos^2 \varphi
\]

\[
C = D_y \cos^2 \varphi + D_z \sin^2 \varphi
\]

The system will then be defined by the powers \(A\), \(B\), \(C\) of each refracting surface, and by the reduced distances \(\delta\) which separate them.

Whereas, in Gauss’s approximation, a centred system is characterized by three independent parameters\[^{[54]}\] [for example, Gauss’s four coefficients subject to relation (108)], to the same approximation, the astigmatic system under consideration needs 10 independent parameters\[^{[55]}\]: let us consider all the incident rays emitted by the point object; each of them has its direction defined by its three direction cosines \(\alpha\), \(\beta\), \(\gamma\) (projections on the coordinate axes of a segment of unit length), these being linked by the relation \(\alpha^2 + \beta^2 + \gamma^2 = 1\); it suffices for \(\beta\) and \(\gamma\) to be given to know the ray\[^{[56]}\]; the corresponding emergent ray\[^{[57]}\] will be characterized in the same way by \(\beta'\) and \(\gamma'\) and the system establishes a relation\[^{[58]}\]

\[
f(\beta, \gamma, \beta', \gamma') = 0
\]

among these quantities. The approximation similar to that of Gauss\[^{[59]}\] consists in retaining, in the development of \(f\), only terms with powers up to and including those of second degree, hence 15 coefficients, that is to say one constant, four terms of first order \((\beta, \gamma, \beta', \gamma')\), four squared terms \((\beta'^2, \gamma^2, \beta'^2, \gamma'^2)\) and six rectangular terms \((\beta\gamma, \text{etc.})\)\[^{[60]}\] as the relation remains valid if one

---

\[^{[53]}\] Comparing these equations with Eqs 30-32 we see that \(C = f_{11}\) and \(B = f_{22}\). Hence \(C\) and \(B\) are the diagonal entries of the dioptric power matrix \(F\) (Eq 29) while \(A\) is twice the off-diagonal entries, \(\text{ie. } F = \begin{pmatrix} C & A/2 \\ A/2 & B \end{pmatrix}\).

This suggests that the expression for \(A\) should be \(A = \frac{1}{2} (D_y - D_z) \sin 2\varphi\) and that the omission of the factor \(\frac{1}{2}\) is a typographical error. LG does not refer to the dioptric power matrix as such but it appears later as the bottom-left submatrix \(\begin{pmatrix} C & A \\ A & B \end{pmatrix}\) of the \(4 \times 4\) matrix for an astigmatic refracting surface.

\[^{[54]}\] Although often stated, as here, that there are 3 independent parameters, that is not strictly correct. If there were 3 independent parameters one would be able to assign arbitrary values to 3 of the parameters. Assigning the value 0 to any 3 of the parameters violates Eq (108) and shows they are not independent. On the other hand, provided that it exists, the principal matrix logarithm of \(\begin{pmatrix} b & d \\ a & c \end{pmatrix}\) does have 3 independent parameters (the off-diagonal entries and 1 of the diagonal entries)\[^{[54]}\].

\[^{[55]}\] A comment similar to Footnote [54] applies here. It is not strictly correct to say that there are 10 independent parameters. Eq (108) is equivalent to saying that \(M\) is symplectic. A symplectic matrix obeys Eq 5. If \(M\) is \(4 \times 4\) then the matrix equation \(M^T EM = E\) is equivalent to 6 equations among the 16 entries. The 6 equations are contained within the 3 matrix equations, Eqs 7-9. Provided it exists the principal matrix logarithm of \(M\) has \(16 - 6 = 10\) independent entries.\[^{[57]}\]

\[^{[56]}\] 2 direction cosines uniquely define the incident segment of a ray from the object point.

\[^{[57]}\] The emergent segment of the ray.

\[^{[58]}\] A ray leaving the object point with particular direction cosines \(\beta\) and \(\gamma\) arrives at the image point with direction cosines \(\beta'\) and \(\gamma'\); \(f\) relates \(\beta, \gamma, \beta', \gamma'\).

\[^{[59]}\] No reference is given in Refs 26 & 27. P 328 of Ref 29 gives the reference ‘Gauss: Dioptrische Untersuchungen (Göttingen 1840)’. See Ref 69.

\[^{[60]}\] For \(\beta\) and \(\gamma\) approaching 0 we approximate the function as a sum of terms up to 2nd degree, \(\text{ie}\) we write

\[
AB^2 + By^2 + CB^2 + D\gamma^2
\]

\[+ E\beta y + F\beta\gamma + G\beta y' + H\gamma\beta' + J\gamma y' + K\beta\gamma' \quad (\text{the six rectangular terms})\]

\[+ L\beta + M\gamma + NB^2 + P\gamma' \quad (\text{the four terms of first order})\]

\[+ Q \quad (\text{the constant})\]

\[= 0.\]

The 15 coefficients are \(A, B, C, D, E, F, G, H, J, K, L, M, N, P, Q\). A ray from an object point through the system obeys this equation. Also the direction cosines become slopes or inclinations. \(\beta\) and \(\gamma\) become the 2 components \(a_{01}\) and \(a_{02}\) of the incident vectorial inclination \(a_0\) and \(\beta'\) and \(\gamma'\) components of the emergent inclination \(a\).
multiplies these 15 coefficients by the same number only 14 of them are independent; but, further, given the recognized symmetry, nothing changes if $\beta$, $\gamma$, $\beta'$ and $\gamma'$ all change sign together which annuls the coefficients of the terms of first order: hence 10 independent parameters.

Just as we defined a centred system by a matrix of second order, we will be able to represent our astigmatic system by a matrix of fourth order, which we will symbolize by $M$, and write:

$$M = \begin{pmatrix} e & j & p & m \\ i & f & n & q \\ d & b & g & k \\ a & c & l & h \end{pmatrix}.$$  

The 16 terms of this matrix include four powers of $(a, b, c, d)$, eight numbers $(e, f, g, h, i, j, k, l)$ and four lengths ($m$, $n$, $p$, $q$); there exist six relations among these terms, which reduce the number of independent coefficients to 10.

One can extend the results obtained above for centred systems to astigmatic systems; we will state them without demonstration for which we refer the reader to the reports by T. Smith (1928).  

1. For a single refracting surface, the origins being coincident with its apex, it becomes:

$$\begin{pmatrix} 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M.$$  

2. Let us displace the origin of the images by a reduced distance $\delta_1$. The system $M_1$ becomes:

$$\begin{pmatrix} 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M.$$  

3. In keeping with the same conventions for the origins for the centred system the linking of two systems $M_1$ and $M_2$ is written:

$$\begin{pmatrix} 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M.$$  

$[61]$. The ray also obeys the same equation if all the coefficients are multiplied by a constant. Eg if $A$ is not 0 we could multiply all the coefficients by $1/ A$; in effect then $A$ reduces to 1 and we are left with 14 coefficients. The same would apply for any other nonzero coefficient.

$[62]$. The system has 2-fold rotational symmetry about the optical axis. Consider rays from an object point on the optical axis. Consider in particular a ray whose incident segment has inclinations $\beta$ and $\gamma$; it has emergent inclinations $\beta'$ and $\gamma'$. Because of the symmetry there is also a ray with incident inclinations $-\beta$ and $-\gamma$ and emergent inclinations $-\beta'$ and $-\gamma'$. This latter ray must also satisfy the equation in Footnote [60]. Thus the equation must still hold if the signs of all the inclinations are changed which is possible only if $L = M = N = P = 0$. 4 more coefficients disappear and we are left with 10.

$[63]$. As for Gaussian optics (Footnote [26]) $M$ is $S^{-1}$. Identifying the entries of $M$ and $S^{-1}$ (Eq 41) gives the relationship of LG’s 16 scalars to entries of the fundamental properties.

$[64]$, $d$, $b$, $a$, $e$ are the 4 entries of the transposed power $F^T$ (see Eq 4). Although $F$ is symmetric for a thin system (see Eq 31) it is not symmetric in general.

$[65]$. $e, j, i, f$ are entries of $D^T$ and $g, k, l, h$ entries of $A^T$.

$[66]$, $p, m, n, q$ are the entries of $-B^T$.

$[67]$. They are the 6 scalar equations contained in matrix Eq 5.


$[69]$. $A, B, C$ are as given above except the missing factor $\frac{1}{2}$ should be inserted in the expression for $A$. See Footnote [53].

$[70]$. The 3rd edn reads “The system becomes $M_1$ such that ...” This does not seem very clear. Displacing the origins of the images by reduced distance $\delta$ is equivalent to considering the compound system of refracting surface followed by homogeneous gap of reduced width $\delta$. If $M_1$ represents the matrix given above for the refracting surface then the matrix for the compound system becomes:

$$\begin{pmatrix} 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = M.$$  

One is multiplying the inverse transference of an astigmatic refracting surface by that of a homogeneous gap:

$$\begin{pmatrix} 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \\ C & A & 1 & 0 \\ A & B & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\delta & 0 \\ 0 & 1 & 0 & -\delta \\ C & A & 1-C\delta & -A\delta \\ A & B & -A\delta & 1-B\delta \end{pmatrix} = M.$$  

Compare Footnote [27].
\[ \mathbf{M}_1 \mathbf{M}_2 = \mathbf{M}. \]

4. For a point object at infinity on axis Ox the condition for correction of astigmatism is written\[ \text{[72]} \]
\[ \frac{l}{a} = \frac{k}{b} = \frac{h}{c} = \frac{g}{d} = x', \]
where \( x' \) designates, as in equation (111), the reduced absolute of the image. This correction holds then for points next to the axis.

5. However the image is not like the object; the transverse magnification\[ \text{[74]} \] in particular varies with direction; to correct the defect the supplementary condition\[ \text{[75]} \]
\[ (113) \]
\[ c = d \quad \text{(or} \quad g = h \) must be realized.

**Application to the eye.** – It can happen that the astigmatic eye depends effectively on 10 independent parameters: this happens if the cornea and the two surfaces of the crystalline lens are all three astigmatic with the axes mutually oblique. Even in this case condition (112) can be verified and the eye corrected by an ordinary thin astigmatic lens: this introduces, in effect, four parameters (its three powers \( A, B \) and \( C \) and its distance to the eye), which will allow the realization of the four equations (112); however, all the possibilities are exhausted and one cannot obtain a magnification independent of the direction.\[ \text{[79]} \] To realize (113) one needs to dispose of one more variable and thus introduce thick correcting systems.\[ \text{[80]} \]

Let us consider the simple case in which the cornea presents the only astigmatism; we will realize the correction with a thick glass composed of two toric surfaces separated by a reduced distance \( \delta_2 \), the second face being a distance \( \delta_4 \) in front of the cornea; taking the principal sections (the same for the three surfaces\[ \text{[81]} \]) as coordinate planes, conditions...
(112) and (113) reduce to \[82\]

\[ B_1 \left[ 1 + (\delta_2 + \delta_4) R_y \right] + B_2 \left[ 1 + (\delta_2 + \delta_4) R_z \right] - B_3 B_3 \left[ \delta_2 + \delta_2 \delta_4 R_y \right] = R_y , \]

\[ C_1 \left[ 1 + (\delta_2 + \delta_4) R_y \right] + C_3 \left[ 1 + (\delta_2 + \delta_4) R_z \right] - C_3 C_3 \left[ \delta_2 + \delta_2 \delta_4 R_y \right] = R_z , \]

\[ C_1 - B_3 \left[ \delta_2 + \delta_4 \right] + \left[ C_3 - B_3 \right] \delta_4 - \left[ C_3 C_3 - B_3 B_3 \right] \delta_4 = 0 \]

where \( R_y \) and \( R_z \) designate the ametropias measured at the corneal apex. There is an infinity of solutions, which allows a supplementary condition to be given. For example, for \( \delta_2 = 4 \text{ mm}, \delta_4 = 2 \text{ mm}, R_y = -7 \text{ dt} \) and \( R_z = -5 \text{ dt} \), one can adopt the following values:

\[ B_1 = +6.38 \text{ dt}^{[84]} \quad B_3 = -14.19 \text{ dt}^{[85]} \quad C_1 = 0 \quad C_3 = -5.32 \text{ dt}. \]

However, as long as one imposes a reasonable limit on the thickness of the correcting glass, it will only be possible in this way to correct those astigmatisms that barely exceed 2 \( \text{dt} \); and in this case the deformations which result from variation of the transverse magnification remain sufficiently weak for it not to be necessary to calculate it.\[87\]

[82]. The compensating system in front of the eye has transference given by Eq 34. For a sharp image it must satisfy Eq 49 or, equivalently, Eq 51. We substitute for \( A_C \) and \( C_C \) from Eq 34 into Eq 49 and make use of Eq 4. After a little rearrangement we arrive at the condition for a sharp image:

\[ \left[ 1 + (\delta_2 + \delta_4) F_0 \right] F_1 + \left[ 1 + (\delta_2 + \delta_4) F_3 - \delta_2 \right] \left[ 1 + (\delta_2 + \delta_4) F_5 \right] F_1 = F_0 . \]  

(57)

This holds in general. We now apply it to the particular example. Because \( \varphi = 0 \) we have for the 1st surface of the lens \( F_1 = \left[ \begin{array}{cc} C_1 & 0 \\ 0 & B_1 \end{array} \right] \) and similarly for the 2nd surface. Also \( F_0 = \left[ \begin{array}{cc} R_y & 0 \\ 0 & R_z \end{array} \right] \) where \( R_y \) is the corneal-plane refractive compensation in the reference meridian and \( R_z \) in the meridian orthogonal to it. All the matrices in Eq 57 are diagonal. Multiplying out we obtain the 2 scalar equations

\[ 1 + (\delta_2 + \delta_4) R_y \left[ C_1 + (\delta_2 + \delta_4) R_y \right] C_3 - \delta_2 \left[ 1 + (\delta_2 + \delta_4) R_y \right] C_3 = R_y , \]

\[ 1 + (\delta_2 + \delta_4) R_z B_1 + \left[ 1 + (\delta_2 + \delta_4) R_z B_3 - \delta_2 \left[ 1 + (\delta_2 + \delta_4) R_z \right] B_3 B_3 \right] = R_z \]

(58)  

(59)

which are identical to the 1st 2 of LG’s 3 equations except that \( y \) and \( z \) are interchanged. His equations are obtained if we choose \( Oy_1 \) to coincide with \( Oz \) instead of \( Oy \) in which case \( \varphi = 90^\circ \). For an undistorted image Eq 55 must also be satisfied. The transference of the cornea is

\[ S_K = \left[ \begin{array}{cc} I & 0 \\ -F_K & I \end{array} \right] \]  

where \( F_K = \left[ \begin{array}{cc} C_K & 0 \\ 0 & B_K \end{array} \right] \). Because there are no astigmatic elements in the rest of the eye the fundamental properties of the rest of the eye are scalar matrices. Thus we can write

\[ S_K = \left[ \begin{array}{cc} A_K & B_K \\ C_K & D_K \end{array} \right] \]

for the transference of the rest of the eye. Hence the transference of the eye is

\[ S = \left[ \begin{array}{cc} A_R & B_R \\ C_R & D_R \end{array} \right] \left[ \begin{array}{cc} I & 0 \\ -F \end{array} \right] = \left[ \begin{array}{cc} A_R - B_R F_K & B_K \bcr C_R - D_R F_K & D_R \end{array} \right] \]

(60)

and so \( B = B_R I \). Hence Eq 55 becomes

\[ I - \delta_2 F_1 - \delta_4 \left( F_3 \left( I - \delta_2 F_1 \right) \right) F_1 = -m B_R R . \]

Again the matrix on the left is diagonal, the diagonal entries being

\[ 1 - \delta_2 C_1 - \delta_4 \left( C_3 \left( I - \delta_2 C_1 \right) + C_1 \right) \]

\[ 1 - \delta_2 B_1 - \delta_4 \left( B_3 \left( I - \delta_2 B_1 \right) + B_1 \right) . \]

Because the off-diagonal entries are 0 Eq 55 can be satisfied in 2 distinct ways: \( \theta = 0^\circ \), in which case \( R = R_{0\theta} = I \), or \( \theta = 45^\circ \), in which case \( R = R_{45^\circ} \). In the 1st case the diagonal entries are equal; their difference is 0 from which we obtain the 3rd of LG’s equations. In the 2nd case their sum is 0 which results in

\[ \left( C_1 + B_1 \delta_2 \right) \delta_4 + \left( C_3 + B_3 \right) \delta_4 - \left( C_3 C_3 - B_3 B_3 \right) \delta_4 = 2 . \]  

(64)

The 2 cases correspond to conditions (a) and (b) respectively (see Condition for sharp undistorted images). Although a theoretical alternative to LG’s 3rd equation this equation leads to absurd results in ordinary applications when used instead of his equation. \( Eq \) if we take LG’s values for \( \delta_2, \delta_4, C_1 \) and use this equation instead of his 3rd equation we obtain the powers \( B_1 = 493.63 \text{ D}, C_3 = -5.32 \text{ D}, B_3 = 498.91 \text{ D} \). Alternatively if \( \delta_2 = \delta_4 = 1 \text{ m}, C_1 = 2 \text{ D} \) we obtain \( B_1 = -0.8 \text{ D}, C_3 = 3.25 \text{ D}, B_3 = 1.5 \text{ D} \). Apparently if widths are reasonable powers are ridiculous and if powers are reasonable widths are ridiculous. This serves to illustrate the fact that condition (b) can be disregarded in practice in ordinary optometric applications.

[83]. There is a typographical error in the original (corrected in the 3rd edn): ‘I y a’ should read ‘Il y a’.

[84]. The original has 5 instead of the correct 8 shown bold here.

[85]. The original has 6 instead of the correct 9 shown bold here.

[86]. By astigmatism LG evidently means \( R_y - R_z \).

[87]. In the 3rd edn LG adds a 1-sentence paragraph: “For these applications of matrix arithmetic to ophthalmic optics one can consult the paper by Miss Bourdy [55].” On p 398 under Bibliographie Ref 27 reads [55]’C. Bourdy, Calcul matriciel et optique paraxiale. Rev. d’Opt., 41 (1962) 295.’ See Ref 72.
when the formula is generalized to linear 0 , Eq 58 on p 108 of LG and Eq 10.4 of LG&EH.

Whence Gullstrand’s notation. ’ The solution is on p 337 of LG.

EXERCISES [p 329]

[p 332]

EXERCISE 43 (Appendix II). – By means of Gauss’s coefficients establish the formulae associated with Gullstrand.

EXERCISE 44 (Appendix II). – The same problem in frontal notation.

SOLUTIONS TO THE EXERCISES [p 333]

[p 341]

43. – If the component systems are referred to their principal points, expression (110) provides us with the resulting matrix

\[
\begin{pmatrix}
1 & 0 & 1 & -\delta & D_1 \\
0 & 1 & 0 & 1 & D_2
\end{pmatrix}
\begin{pmatrix}
b \\ d
\end{pmatrix}
= \begin{pmatrix}
a \\ c
\end{pmatrix},
\]

which gives

\[
a = D_1 + D_2 - \delta D_1 D_2 ,
\]

\[
b = 1 - \delta D_2 ,
\]

\[
c = 1 - \delta D_1 ,
\]

\[
d = -\delta .
\]

We have seen that the resulting power was a, and that the reduced abscissas of the principal points are written \((1 - b)/a\) and \((c - 1)/a\). Whence Gullstrand’s formulae.[88]

44. – In designating the abscissa in the first system by \(x_0\) from the resulting object origin one obtains[89]

\[
\begin{pmatrix}
g'_1 \\
g'_2
\end{pmatrix}
= \begin{pmatrix}
g'_1 - \xi g'_2 \\
g'_1 \xi g'_2
\end{pmatrix}
= \begin{pmatrix}
\frac{1}{1 - x_0 g' / g'} \\
1
\end{pmatrix}
\begin{pmatrix}
g'_1 / g'_2 \\
1 / g'_2
\end{pmatrix}.
\]

Whence the classical formulae by identification (see exercise 22).[90]

References


[88] On p 343 of his Bibliographie LG gives the reference ‘A. Gullstrand, Einführung in die Methoden de Dioptrik des Auges des Menschen. (Leipzig, 1911)’. The formula for \(a\) is the formula often referred to as Gullstrand’s formula. Notice the order of multiplication \(D_2 D_1\); it is correct here (in Gaussian optics) but must be reversed to \(D_2 D_1\) when the formula is generalized to linear optics.[10, 73-75]

[89] Multiplication results in

\[
\begin{pmatrix}
g'_1 (g'_2 - \xi D'_2 / g'_2) \\
g'_1 \xi D'_2 / g'_2
\end{pmatrix}
= \begin{pmatrix}
g'_1 - \xi (D'_1 / g'_1) \\
g'_1 \xi (D'_1 / g'_1)
\end{pmatrix}
= \begin{pmatrix}
g'_1 - x_0 D'/g' - x_0 / g' \\
D'/g' - 1 / g'
\end{pmatrix}.
\]

Equating the bottom-right entries and rearranging we obtain the shape factor

\[
g' = \frac{g' \xi D'_1}{1 - \xi D'},
\]

which is the equation near the bottom of p 108 of LG and Eq 10.5 of LG&EH. From the bottom-left entries and Eq 65 we obtain the back-vertex power (frontal rear power in LG&EH) \(D' = g'_2 \xi D'_1 / (1 - \xi D'),\) Eq 58 on p 108 of LG and Eq 10.4 of LG&EH.

[90] Ex 22 (on p 330 of LG) reads: ‘By means of homography examine the combination of two centred systems in frontal notation.’ The solution is on p 337 of LG.


27. Le Grand Y. *Optique Physiologique. Tome Premier. La Dioptrie de l’Œil et Sa Correction* (3rd ed). Paris: Masson 1964 374-381. [WFH’s copy gives the publisher the same as that for the first edition (see Ref 26) and carries the date 1965 on the cover and 1964 on the title page. However the publisher’s name has been pasted over with the new name (Masson).]


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