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Inner-product spaces for quantitative analysis of eyes and other optical systems

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Because dioptric power matrices of thin systems constitute a (three-dimensional) inner-product space, it is possible to define distances and angles in the space and so do quantitative analyses on dioptric power for thin systems. That includes astigmatic corneal powers and refractive errors. The purpose of this study is to generalise to thick systems. The paper begins with the ray transference of a system. Two 10-dimensional inner-product spaces are devised for the holistic quantitative analysis of the linear optical character of optical systems. One is based on the point characteristic and the other on the angle characteristic; the first has distances with the physical dimension L^{-1} and the second has the physical dimension L. A numerical example calculates the locations, distances from the origin and angles subtended at the origin in the 10-dimensional space for two arbitrary astigmatic eyes.

Introduction

The optical character of a thin system in linear optics can be represented by a symmetric 2×2 matrix **F**, the symmetric dioptric power matrix. The set of all such powers defines a three-dimensional linear (or vector) space, known as symmetric dioptric power space.[1] Because the matrix has uniform physical dimensionality[2] (each entry has the dimension L^{-1} and is usually measured in dioptres), one can define an inner-product on the space and the space becomes an inner-product space. Because symmetric dioptric power space is an inner-product space, we have been able to define distances, angles, orthonormal axes, confidence ellipsoids, etc. in the space. This has provided the basis for the quantitative analysis we have done on powers including refractive errors and corneal powers (e.g. Ref 3).

For some years, we have sought to extend this type of analysis to thick systems such as the eye (e.g. Ref 4). In linear optics, the optical character of a system that is thick or thin is completely characterised by the ray transference (a real 4×4 matrix)

$$\mathbf{S} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix} \qquad \text{[Eqn 1]}$$

of the system.[5] In strong contrast to the set of symmetric dioptric powers, the set of transferences is neither a linear space nor does it have uniform dimensionality. Therefore, there is no inner-product space that would provide a basis for holistic quantitative analysis of the optical character of thick systems such as the eye. The purpose of this study is to show how inner-product spaces can in fact be constructed for general optical systems.

Method

The method is based on the transference. The transference **S** (Equation 1) obeys the following equation[5]

$$\mathbf{S}^{\mathrm{T}}\mathbf{E}\mathbf{S} = \mathbf{E} \qquad \text{[Eqn 2]}$$

where

$$\mathbf{E} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix} \qquad \text{[Eqn 3]}$$

and **I** and **O** are identity and null matrices, respectively. Such matrices are called *symplectic*.[6] **A**, **B**, **C** and **D** are 2×2 submatrices of **S** and represent the *fundamental* (linear) optical properties of the system.[7] **B** has the physical dimension L and **C** the physical dimension L^{-1}; the other two fundamental properties are dimensionless. Other optical properties of the system can be obtained from the fundamental properties; for example, the power of the system is given by[7]

$$\mathbf{F} = -\mathbf{C} \qquad \text{[Eqn 4]}$$

and, for eyes, the corneal-plane refractive compensation (or refractive error) is given by[7]

$$\mathbf{F}_0 = \mathbf{B}^{-1}\mathbf{A}. \qquad \text{[Eqn 5]}$$

Two matrices related to the transference are the *point characteristic*

$$\mathbf{P} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1} \\ -\mathbf{B}^{-\mathrm{T}} & \mathbf{D}\mathbf{B}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{V}^{\mathrm{T}} & \mathbf{W} \end{pmatrix} \qquad \text{[Eqn 6]}$$

and the *angle characteristic*

$$\mathbf{Q} = \begin{pmatrix} \mathbf{C}^{-1}\mathbf{D} & \mathbf{C}^{-1} \\ \mathbf{C}^{-\mathrm{T}} & \mathbf{A}\mathbf{C}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{X} & \mathbf{Y} \\ \mathbf{Y}^{\mathrm{T}} & \mathbf{Z} \end{pmatrix}. \qquad \text{[Eqn 7]}$$

Elsewhere[8] we use these matrices to calculate average systems.

Results

From **P** and **Q** we construct the 2×6 matrices

$$\mathbf{G} = (\mathbf{U} \quad \mathbf{V} \quad \mathbf{W}) \qquad \text{[Eqn 8]}$$

and

$$\mathbf{H} = (\mathbf{X} \quad \mathbf{Y} \quad \mathbf{Z}). \qquad \text{[Eqn 9]}$$

It is a consequence of symplecticity (Equation 2) that **U**, **W**, **X** and **Z** are symmetric; **V** and **Y** are general. (Properties of symplectic matrices are summarised elsewhere.[9]) The set of all matrices **G** is a linear space and **G** has uniform physical dimensionality (L^{-1}). Similarly matrices **H** define a dimensionally uniform (dimension L) linear space.

G can be expanded as

$$\begin{aligned}\mathbf{G} = {} & U_{\mathbf{I}}\begin{pmatrix}\mathbf{I} & \mathbf{O} & \mathbf{O}\end{pmatrix} + U_{\mathbf{J}}\begin{pmatrix}\mathbf{J} & \mathbf{O} & \mathbf{O}\end{pmatrix} + U_{\mathbf{K}}\begin{pmatrix}\mathbf{K} & \mathbf{O} & \mathbf{O}\end{pmatrix} \\ & + V_{\mathbf{I}}\begin{pmatrix}\mathbf{O} & \mathbf{I} & \mathbf{O}\end{pmatrix} + V_{\mathbf{J}}\begin{pmatrix}\mathbf{O} & \mathbf{J} & \mathbf{O}\end{pmatrix} + V_{\mathbf{K}}\begin{pmatrix}\mathbf{O} & \mathbf{K} & \mathbf{O}\end{pmatrix} + V_{\mathbf{L}}\begin{pmatrix}\mathbf{O} & \mathbf{L} & \mathbf{O}\end{pmatrix} \\ & + W_{\mathbf{I}}\begin{pmatrix}\mathbf{O} & \mathbf{O} & \mathbf{I}\end{pmatrix} + W_{\mathbf{J}}\begin{pmatrix}\mathbf{O} & \mathbf{O} & \mathbf{J}\end{pmatrix} + W_{\mathbf{K}}\begin{pmatrix}\mathbf{O} & \mathbf{O} & \mathbf{K}\end{pmatrix}\end{aligned} \qquad \text{[Eqn 10]}$$

where

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Also,

$$\begin{aligned} V_{\mathbf{I}} &= (v_{11} + v_{22})/2 \qquad V_{\mathbf{J}} = (v_{11} - v_{22})/2 \\ V_{\mathbf{K}} &= (v_{12} + v_{21})/2 \qquad V_{\mathbf{L}} = (v_{12} - v_{21})/2 \end{aligned} \qquad \text{[Eqn 11]}$$

and similarly for $U_{\mathbf{I}}$ and the other coefficients in Equation 10. We define the coordinate vector

$$\mathbf{g} = \begin{pmatrix} U_{\mathbf{I}} & U_{\mathbf{J}} & U_{\mathbf{K}} & V_{\mathbf{I}} & V_{\mathbf{J}} & V_{\mathbf{K}} & V_{\mathbf{L}} & W_{\mathbf{I}} & W_{\mathbf{J}} & W_{\mathbf{K}} \end{pmatrix}^{\mathrm{T}} \qquad \text{[Eqn 12]}$$

relative to the basis

$$\beta = \left\{ \begin{pmatrix}\mathbf{I} & \mathbf{O} & \mathbf{O}\end{pmatrix}, \begin{pmatrix}\mathbf{J} & \mathbf{O} & \mathbf{O}\end{pmatrix}, \begin{pmatrix}\mathbf{K} & \mathbf{O} & \mathbf{O}\end{pmatrix}, \cdots, \begin{pmatrix}\mathbf{O} & \mathbf{O} & \mathbf{K}\end{pmatrix} \right\}. \qquad \text{[Eqn 13]}$$

Consider two optical systems 1 and 2. Their coordinate vectors are $\mathbf{g}_1$ and $\mathbf{g}_2$. Now we define the inner-product of $\mathbf{g}_1$ and $\mathbf{g}_2$ by

$$\langle \mathbf{g}_1, \mathbf{g}_2 \rangle = \mathbf{g}_1^{\mathrm{T}} \mathbf{g}_2 \ . \qquad \text{[Eqn 14]}$$

Consequently we have distances (magnitudes) g and angles θ in the space defined by

$$g = \sqrt{\mathbf{g}^{\mathrm{T}}\mathbf{g}} \qquad \text{[Eqn 15]}$$

and

$$\cos\theta = \frac{\mathbf{g}_1^{\mathrm{T}}\mathbf{g}_2}{g_1 g_2} \qquad \text{[Eqn 16]}$$

respectively.

Thus we have a 10-dimensional inner-product space for quantitative analysis of optical systems in linear optics for which **B** is non-singular. One can think of distances in the space as powers (e.g. dioptres).

For matrices of the form **H** (Equation 9), one can follow a similar approach. It leads to a second 10-dimensional inner-product space. It applies for optical systems for which **C** is non-singular and distances in the space are lengths (e.g. metres).

We illustrate the theory using two optical systems whose transferences have been presented before:[8]

$$\mathbf{S}_1 = \begin{pmatrix} -0.2066 & -0.0031 & 0.0200 & 0.0000 \\ -0.0031 & -0.2240 & 0.0000 & 0.0200 \\ -58.8160 & -0.0853 & 0.8569 & 0.0017 \\ -0.0841 & -59.5090 & 0.0017 & 0.8599 \end{pmatrix}$$

and

$$\mathbf{S}_2 = \begin{pmatrix} -0.1641 & 0.0060 & 0.0197 & 0.0000 \\ 0.0060 & -0.1399 & 0.0000 & 0.0197 \\ -57.9190 & 0.3455 & 0.8670 & 0.0024 \\ 0.3415 & -56.9734 & 0.0024 & 0.8637 \end{pmatrix},$$

the units being dioptres and metres. The coordinate vectors (Equation 12) turn out to be

$$\mathbf{g}_1 = (-10.7650 \quad 0.4350 \quad -0.1550 \quad -50.0000 \quad 0 \quad 0 \quad 0 \quad 42.9200 \quad -0.0750 \quad 0.0850)^{\mathrm{T}}\,\mathrm{D}$$

and

$$\mathbf{g}_2 = (-7.7157 \quad -0.6142 \quad 0.3046 \quad -50.7614 \quad 0 \quad 0 \quad 0 \quad 43.9264 \quad 0.0838 \quad 0.1218)^{\mathrm{T}}\,\mathrm{D}$$

These vectors locate the two optical systems relative to the origins of the space. Their distances from the origin are $g_1 = 66.77$ D and $g_2 = 67.57$ D, respectively, and they subtend an angle $\theta = 2.90°$ at the origin.

Conclusion

We have here constructed two inner-product spaces for the linear optical characters of optical systems. One is based on the point characteristic and the other on the angle characteristic. Both spaces can be used for eyes because they have non-singular **B** and **C**. We now have the machinery for holistic quantitative analysis of optical systems in general and eyes in particular.

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Competing interests

The authors declare that they have no financial or personal relationships which may have inappropriately influenced them in writing this article.

Authors' contributions

The work was a team effort led by W.F.H. with contributions from T.E. and R.D.v.G. over several years.

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