Target refractions in bilateral monofocal pseudophakia: analysis of a mathematical model

William Frith Harris*, Finn Kristian Næser[†] and Jesper Østergaard Hjortdal[§]

Department of Optometry, University of Johannesburg, P O Box 524, Auckland Park 2006, South Africa

†Department of Ophthalmology, Randers Regional Hospital, 8930 Randers NØ, Denmark

§Department of Ophthalmology, Aarhus University Hospital, 8000 Aarhus C, Denmark

<wharris@uj.ac.za>   <krisnaes@rm.dk>   <jesper.hjortdal@dadlnet.dk>

Abstract

A recent paper proposes a model for suitable target refractions for monofocal pseudophakic monovision. The purpose of this paper is to examine the mathematics that underpins the theory. The expressions for the optimal target refractions presented before are shown formally to define the minimum of the mean binocular refractive error over a range of object distances. The model is shown to be much more sensitive to errors in the distal target refraction than to errors in the proximal refraction. It is also more sensitive to the choice of proximal limit of the object range than to the distal limit. These considerations are important in selecting and evaluating suitable combinations of refractions in pseudophakic monovision. (*S Afr Optom* 2013 **72**(3) 103-109)

Key words: Target refraction, pseudophakia, monovision, mathematical model, astigmatism

Introduction

The aim of contemporary cataract surgery is to obtain optimal uncorrected visual acuity over the most extended fixation range. This may be achieved by the system of monovision where one eye is targeted for distance and the other eye for near.[1, 2] Monovision has been widely used with contact lenses. In younger presbyopes some residual natural lens accommodation remains in eyes fitted with contact lenses and the procedure is essentially reversible.[3] On the other hand there is essentially no accommodation in the pseudophakic eye, and any change in refraction will require new surgical procedures[4, 5]. It is therefore important to establish appropriate target refractions for the far and near eye, and to analyse the impact of deviations from these targets.

A simple model has recently been described by Naeser and coworkers[6] for defining suitable target refractions F_D and F_P for each eye following surgery for bilateral monofocal pseudophakia. Subscripts D and P suggest distal and proximal respectively. The model leads to the optimal target refractions

$$F_{Dm} = \frac{2}{z_D - \sqrt{z_D z_P}}, \tag{1}$$

the distal refraction for one eye, and

$$F_{Pm} = \frac{2}{z_P - \sqrt{z_D z_P}}, \tag{2}$$

the proximal refraction for the other eye. z_D and z_P are the axial positions defined in Figure 1; they

represent a range in front of the eye within which object points are supposed to be visible with minimal blur. They are measured in the direction opposite to the direction of light which is defined by longitudinal axis Z and are negative. We set

$$z_D < z_P < 0. \tag{3}$$

Because hyperopic refractions would induce blur for both distant and proximal fixation we take the target refractions to be myopic and set
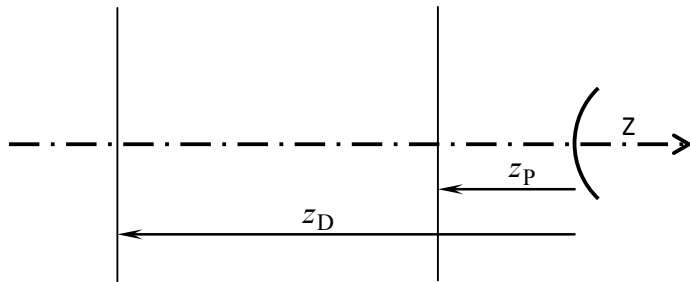
$$F_P < F_D < 0. \tag{4}$$

Figure 1 Range of positions in front of the eye in which objects should be seen with minimal blur. z_D and z_P represent the distal and proximal limits of the range. Both are negative.

The purpose of this paper is to take a closer look at the mathematics underlying the theory. We examine the nature of the mean binocular error as a function of the four variables F_D, F_P, z_D and z_P and show formally that Equations 1 and 2 do indeed represent a global minimum. We also examine the nature of the mean binocular error in the neighbourhood of the minimum. It is convenient here to use a mathematical notation which differs from that used originally[6].

The surgeon's objective is to give the patient optimal vision for objects in the range from a distant transverse plane at axial position z_D to a proximal transverse plane at axial distance z_P (Figure 1). What refractions F_D and F_P should he or she aim for in the two eyes? Figure 1 suggests that z_D and z_P are measured from the cornea; they could also be measured from the spectacle plane. The refractions in those cases are corneal- and spectacle-plane refractions respectively.

Pseudophakic eye

Implicit in the discussion so far is that the refractions of the two pseudophakic eyes should be stigmatic,

that is, F_D and F_P should be real scalars. However there are advocates of astigmatic refractions[7-11]. In order to allow for that possibility we work instead with the 2x2 dioptric power matrix **F** instead of the scalar F. Scalar target refractions F_D and F_P are replaced by matrix target refractions $\mathbf{F}_D$ and $\mathbf{F}_P$. As explained elsewhere[12] **F** can be expanded as

$$\mathbf{F} = \mathbf{I}F_I + \mathbf{J}F_J + \mathbf{K}F_K \tag{5}$$

where **I** is an identity matrix,

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6}$$

$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{7}$$

and F_I, F_J and F_K are scalar coefficients of the power **F**. F_I represents the scalar component; it is identical to what Naeser and Hjortdal[6, 13, 14] represent by SEP and what is commonly called the equivalent sphere. F_J and F_K account for the astigmatism; they are essentially equivalent to what Naeser and Hjortdal[14] represent by KP(90) and KP(135) and Thibos and coworkers[15] by J_0 and J_{45} respectively. The magnitude of the power **F** is[12]

$$F = \sqrt{F_I^2 + F_A^2} \tag{8}$$

where

$$F_A = \sqrt{F_J^2 + F_K^2} \tag{9}$$

can be thought of as the magnitude of the astigmatism. In particular the coefficients of $\mathbf{F}_D$ are F_{DI}, F_{DJ} and F_{DK} and the two magnitudes are F_D and F_{DA} and similarly for $\mathbf{F}_P$. Thus the surgeon's objective is rephrased as, for what possibly astigmatic distal and proximal refractions $\mathbf{F}_D$ and $\mathbf{F}_P$ should he or she aim in order to obtain optimal visual acuity in the fixation interval from z_D to z_P?

Consider a pseudophakic eye of refraction **F** which views an object in a transverse plane at axial position z. If it so happens that $\mathbf{F} = \mathbf{I}/z$ there will be no blur. In general there is an error

$$\mathbf{F}_e = \mathbf{F} - \mathbf{I}/z \tag{10}$$

of magnitude

$$F_e = \sqrt{(F_I - 1/z)^2 + F_A^2} \tag{11}$$

which we take to be a measure of the blur in that eye when it views the object at axial position z. (Elsewhere[6] it is called the defocus.) It is evident from Equation 11 that the error is minimized if $F_A^2 = 0$. This implies $F_J = F_K = 0$ in Equation 9 and, hence, that the refraction is $\mathbf{F} = \mathbf{I}F_I$ and, hence, is purely stigmatic. It follows that the refraction can be completely characterized by the scalar F_I and that we can drop matrices from the discussion and revert to scalars. Furthermore the subscript on F_I is redundant and we can write F instead. Thus Equation 11 reduces to

$$F_e = \sqrt{\left(F - 1/z\right)^2} . \tag{12}$$

We shall refer to F_e simply as the (monocular) error.

Below we shall need the integral of the error F_e over the range z_D to z_P, namely

$$\int_{z_D}^{z_P} F_e \, dz = \left(z_P + z_D\right)F - \log z_P z_D F^2 - 2 . \tag{13}$$

(Elsewhere[6] it is called the cumulative defocus.)

Written in particular for the eye with refraction F_D Equation 12 becomes

$$F_{De} = \sqrt{\left(F_D - 1/z\right)^2} \tag{14}$$

and similarly for the eye with refraction F_P and error F_{Pe}. The two errors F_{De} (blue) and F_{Pe} (red) vary with z as illustrated in Figure 2. Figure 2 is drawn for $F_P = -0.27$ D and $F_P = -1.15$ D, the refractions calculated before[6]. The two error curves intersect in the point with axial position z_χ and error $F_{e\chi}$. Hence, from Equation 14 and its counterpart for F_{Pe}, we obtain

$$\sqrt{\left(F_D - 1/z_\chi\right)^2} = \sqrt{\left(F_P - 1/z_\chi\right)^2} \tag{15}$$

or

$$F_D - 1/z_\chi = \pm\left(F_P - 1/z_\chi\right). \tag{16}$$

The plus sign corresponds to the case $F_D = F_P$ which is disallowed by Inequality 4. We are left with

$$F_P - \frac{1}{z_\chi} = -F_D + \frac{1}{z_\chi} . \tag{17}$$

and, hence,

$$z_\chi = \frac{2}{F_D + F_P} \tag{18}$$

or

$$\frac{1}{z_\chi} = \frac{F_D + F_P}{2} . \tag{19}$$

Thus the two error curves intersect where the inverse of the axial position is simply the arithmetic average of the two refractions.
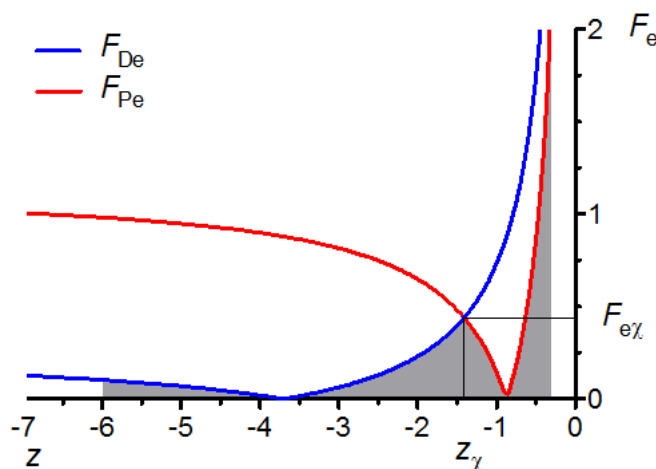

Figure 2   The monocular errors F_{De} (blue) and F_{Pe} (red) for refractions $F_D = -0.27$ D and $F_P = -1.15$ D. (Modified from Reference 6.)

Substitution from Equation 19 into Equation 14 gives the error at the intersection,

$$F_{e\chi} = \frac{F_D - F_P}{2} . \tag{20}$$

Thus the inverse of the axial position of the intersection in Figure 2 is the semi-sum (Equation 19) and the error there is the semi-difference (Equation 20) of the two refractions. In particular for Figure 2 $\frac{1}{z_\chi} = -0.17$ D and $F_{e\chi} = 0.44$ D approximately.

Binocular error

In agreement with the assumption made by Naeser and coworkers[6] we now define the binocular error F_{be}

as the lower of the two monocular errors. Thus

$$F_{be} = \begin{cases} F_{De} \text{ for } z \le z_\chi \\ F_{Pe} \text{ for } z > z_\chi \end{cases}, \qquad (21)$$

as is clear from Figure 2. The *mean binocular error* over the range of object positions from z_D to z_P is

$$\overline{F}_{be} = \frac{1}{z_P - z_D} \int_{z_D}^{z_P} F_{be} \, dz \qquad (22)$$

which, making use of Equation 21, we can split into the sum of two integrals as

$$\overline{F}_{be} = \frac{1}{z_P - z_D} \left(\int_{z_D}^{z_\chi} F_{De} dz + \int_{z_\chi}^{z_P} F_{Pe} dz \right). \qquad (23)$$

Each integral is of the form of Equation 13. Hence, applying Equation 13 twice we obtain

$$\overline{F}_{be} = \frac{1}{z_P - z_D} \Big(\big(z_\chi + z_D\big)F_D - \log z_\chi z_D F_D^2 - 2 \cdot$$

$$+ \big(z_P + z_\chi\big)F_P - \log z_P z_\chi F_P^2 - 2 \Big) \qquad (24)$$

which with substitution from Equation 18 simplifies to

$$\overline{F}_{be} = \frac{1}{z_P - z_D} \left(z_D F_D + z_P F_P + \log \frac{\left(\dfrac{1}{F_P} + \dfrac{1}{F_D}\right)^2}{4 z_P z_D} - 2 \right). \qquad (25)$$

Equation 25 represents the mean binocular error as a function of the four independent variables, the two refractions F_D and F_P and the extremes of the object range z_D and z_P.

Equation 25 is plotted as a surface in Figure 3 for $z_D = -6$ m and $z_P = -0.33$ m for a range of values of F_P and a range of values of F_D. The height of the surface represents the mean binocular error $\overline{F}_{be}$ for the particular values of z_D, z_P, F_D and F_P. Figure 4 is a contour plot of the surface. The dot locates the minimum mean binocular error $\overline{F}_{bem} = 0.1712$ D. The contours are at intervals of 0.025 D. The innermost contour is at 0.175 D. Towards the top left the surface rises steeply; contours there at levels greater than 0.5 D have been omitted. As these figures show the surface has the form of a relatively long and narrow valley with a single minimum.
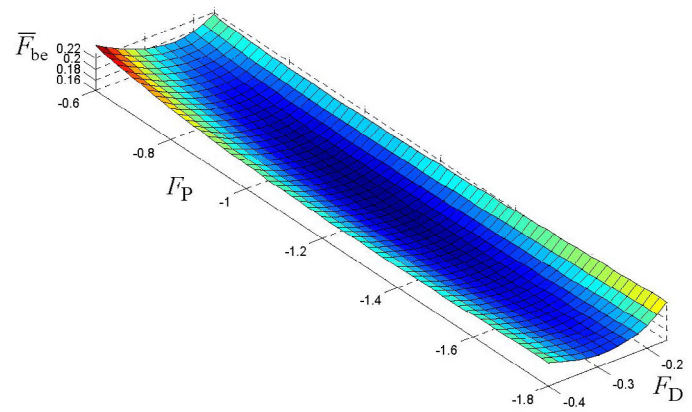

Figure 3   The mean binocular error $\overline{F}_{be}$ as a function of the proximal F_P and distal F_D refractions for a region around the minimum, the units being dioptres. The object range is $z_D = -6$ m to $z_P = -0.33$ m.
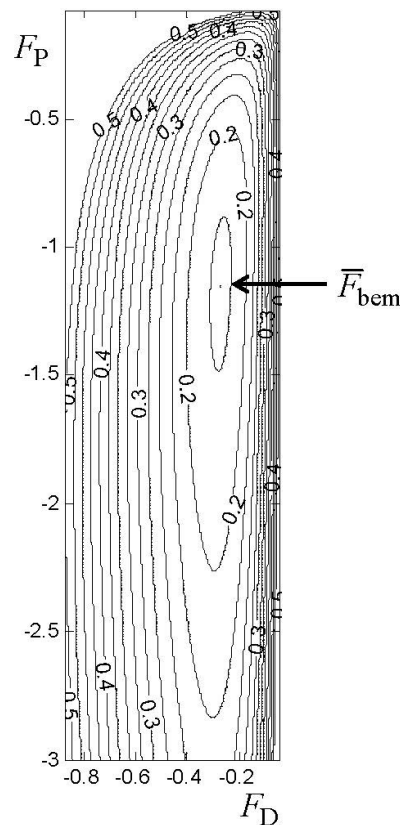

Figure 4   A contour plot of the mean binocular error $\overline{F}_{be}$ as a function of the proximal F_P and distal F_D refractions for the object range $z_D = -6$ to $z_P = -0.3$ m. The minimum mean binocular error, $\overline{F}_{bem} = 0.1712$ D, is represented by the dot, the corresponding optimum target refractions being $F_{Pm} = -0.1513$ and $F_{Dm} = -0.2700$ D. The contours are at intervals of 0.025 D. The innermost contour is at 0.175 D. Towards the top left the surface rises steeply; contours there at levels greater than 0.5 D have been omitted.

106

Dependence of the mean binocular error on the two refractions

From Equation 25 we obtain the two derivatives

$$\frac{\partial \overline{F}_{be}}{\partial F_D} = \frac{1}{z_P - z_D}\left(z_D - \frac{2}{F_D\left(\dfrac{F_D}{F_P}+1\right)}\right) \tag{26}$$

and

$$\frac{\partial \overline{F}_{be}}{\partial F_P} = \frac{1}{z_P - z_D}\left(z_P - \frac{2}{F_P\left(\dfrac{F_P}{F_D}+1\right)}\right). \tag{27}$$

We are seeking values of refractions F_D and F_P that minimize the mean binocular error $\overline{F}_{be}$ (given by Equation 25) over a specified object range from z_D to z_P. They are the optimal target refractions which we represent by F_{Dm} and F_{Pm}. A necessary condition that $\overline{F}_{be}$ has a minimum at $F_D = F_{Dm}$ and $F_P = F_{Pm}$ is that the derivatives of Equations 26 and 27 vanish. This gives us two equations in the two unknowns F_{Dm} and F_{Pm}. Solving them we find that

$$F_{Dm} = \frac{2}{z_D \pm \sqrt{z_D z_P}} \tag{28}$$

and

$$F_{Pm} = \frac{2}{z_P \pm \sqrt{z_D z_P}}. \tag{29}$$

Inequality 3 implies that $\sqrt{z_D z_P} > |z_P|$. The plus sign in the above two equations then implies $F_{Pm} > 0$ in violation of Inequality 4 and must, therefore, be dropped from both equations. Hence we obtain Equations 1 and 2.

Equations 1 and 2 are necessary but not sufficient conditions for a minimum in binocular error. To ensure they represent a minimum we need to determine second derivatives. They turn out to be

$$\frac{\partial^2 \overline{F}_{be}}{\partial F_D^2} = \frac{2F_P(2F_D + F_P)}{(z_P - z_D)(F_D + F_P)^2 F_D^2} \tag{30}$$

$$\frac{\partial^2 \overline{F}_{be}}{\partial F_P^2} = \frac{2F_D(2F_P + F_D)}{(z_P - z_D)(F_P + F_D)^2 F_P^2} \tag{31}$$

and

$$\frac{\partial^2 \overline{F}_{be}}{\partial F_P \partial F_D} = -\frac{2}{(z_P - z_D)(F_D + F_P)^2}. \tag{32}$$

Then

$$\frac{\partial^2 \overline{F}_{be}}{\partial F_D^2}\frac{\partial^2 \overline{F}_{be}}{\partial F_P^2} - \left(\frac{\partial^2 \overline{F}_{be}}{\partial F_P \partial F_D}\right)^2$$

$$= \frac{8}{F_D F_P (z_P - z_D)^2 (F_D + F_P)^2}. \tag{33}$$

Because of Inequalities 3 and 4 the right-hand sides of Equations 30, 31 and 33 are all greater than zero. Thus the sufficient condition[16] is satisfied and, hence, Equations 1 and 2 do indeed define a unique minimum mean binocular error $\overline{F}_{bem}$ over the specified range of object positions.

Evaluated at the minimum of surfaces of the type shown in Figures 3 and 4 Equations 30 and 31 give the curvature of the surface at the minimum in a direction parallel to the F_D and F_P axes respectively. (A positive curvature means that moving along the surface from the minimum takes one upward; a negative curvature would imply a maximum instead of the minimum we have here.)

For $z_D = -6$ m and $z_P = -0.33$ m Equations 1 and 2 give optimum refractions $F_{Dm} = -0.2700$ D and $F_{Pm} = -1.1513$ D. The binocular error is zero at axial distances $1/F_{Dm} = -3.7036$ m and $1/F_{Pm} = -0.8686$ m. For objects at axial distances $z_D = -6$ m and $z_P = -0.33$ m the binocular errors are 0.1033 D and 1.8790 D respectively. The latter figure represents the maximum binocular error for the object range and is at the proximal point of the range. With these optimal refractions the axial position for error equality of the two eyes is (Equation 18) $z_\chi = -1.4071$ m, the binocular error there being (Equation 14) $F_\chi = 0.4407$ D. The mean binocular error for the object range (Equation 25) is $\overline{F}_{be} = 0.1712$ D. With these refractions Equations 30 and 31 result in $\partial^2 \overline{F}_{be}/\partial F_D^2 = 4.6636$ m and $\partial^2 \overline{F}_{be}/\partial F_P^2 = 0.0915$

m. Thus the curvature at the minimum in the F_D direction is more than 50 times the curvature in the F_P direction. There is, therefore, much greater tolerance for errors in the proximal target refraction F_{Pm} than in the distal target refraction F_{Dm}.

Dependence of the mean binocular error on the object range

From Equation 25 we also obtain the derivatives

$$\frac{\partial \overline{F}_{be}}{\partial z_D} = \frac{1}{(z_P - z_D)^2} \left(z_P \left(F_P + F_D - \frac{1}{z_D} \right) + \log \frac{\left(\frac{1}{F_P} + \frac{1}{F_D} \right)^2}{4 z_P z_D} - 1 \right)$$

(34)

and

$$\frac{\partial \overline{F}_{be}}{\partial z_P} = -\frac{1}{(z_P - z_D)^2} \left(z_D \left(F_P + F_D - \frac{1}{z_P} \right) + \log \frac{\left(\frac{1}{F_P} + \frac{1}{F_D} \right)^2}{4 z_P z_D} - 1 \right).$$

(35)

For the object range $z_D = -6$ m to $z_P = -0.33$ m and the optimal refractions calculated above one obtains $\partial \overline{F}_{be} / \partial z_D = 0.01196$ D/m and $\partial \overline{F}_{be} / \partial z_P = 0.30120$ D/m. Thus the sensitivity of the mean binocular error to small change in the proximal limit of the object range is nearly 30 times that in the distal limit.

Concluding remarks

The analysis here provides formal proof of the result presented before[6] that the optimal target refractions F_{Dm} and F_{Pm} for monofocal pseudophakia over an object range from z_D to z_P are given by Equations 1 and 2. The optimal target refractions locate the minimum of the mean binocular error over the object range for distal F_D and proximal F_P target refractions. Figures 3 to 4 show the mean binocular error graphically for the object range from $z_D = -6$ to $z_P = -0.33$ m. The surface is a long narrow valley with a minimum at the optimal refractions $F_{Dm} = -0.2700$ D and $F_{Pm} = -1.1513$ D. Because

the curvature at the bottom of the valley in the F_D direction is some 50 times the curvature in the F_P direction it means that a small change in distal refraction from the optimum takes one up the sides of the valley much faster than does a small change in proximal refraction. This suggests that errors in the distal refraction are considerably more important than errors in the proximal refraction. Thus small deviations of the proximal refraction from the target refraction have limited impact on the mean binocular error while there may be a need for additional surgery where distal refractions are off target. On the other hand, around the optimum, the mean binocular error is some 30 times more sensitive to the proximal limit z_P of the axial range than to the distal limit z_D. Thus the choice of the proximal limit is more important than the choice of the distal limit. It may be important, therefore, to tailor the proximal target for the individual patient.

Acknowledgements

References

1. Evans BJW. Monovision: a review. *Ophthal Physiol Opt* 2007 **27** 417-439.
2. Johannsdottir KR, Stelmach LB. Monovision: a review of the scientific literature. *Optom Vis Sci* 2001 **78** 646-651.
3. Gutkowski M, Cassin B. Stereopsis and monovision in the contact lens management of presbyopia. *Bin Vis Q* 1990 **6** 31-36.
4. Finkelman YM, Ng JQ, Barrett GD. Patient satisfaction and visual function after pseudophakic monovision. *J Cataract Refract Surg* 2009 **35** 998-1002.
5. Greenbaum S. Monovision pseudophakia. *J Cataract Refract Surg* 2002 **28** 1439-1443.
6. Naeser K, Hjortdal J, Harris WF. Pseudophakic monovision: optimal distribution of refractions. *Acta Ophthalmol* 2013 May 7. doi: 10.1111/aos.12148. [Epub ahead of print]
7. Huber C. Planned myopic astigmatism as a substitute for accommodation in peudophakia. *Am Intraoc Implant Soc J* 1981 **7** 244-249.
8. Datiles MB, Gancayco T. Low myopia with low astigmatism correction gives cataract surgery patients good depth of focus. *Ophthalmol* 1990 **97** 922-926.
9. Sawusch MR, Guyton DL. Optimal astigmatism to enhance

depth of focus after cataract surgery. *Ophthalmol* 1991 **98** 1025-1029.

10. Bradbury JA, Hilman JS, Cassells-Brown A.   Optimal postoperative refraction for good unaided near and distance vision with monofocal intraocular lenses. *Br J Ophthalmol* 1992 **76** 300-302.

11. Trindade F, Oliveira A, Frasson M.   Benefit of against-the-rule astigmatism to uncorrected near acuity. *J Cataract Refract Surg* 1997 **23** 82-85.

12. Harris WF.   Optimal target refraction for implantation of monofocal intraocular lenses. *Acta Ophthalmol* 2012 **90** 75-76.

13. Naeser K, Hjortdal, J.   Optimal refraction with monofocal intraocular lenses: no beneficial effects of astigmatism. *Acta Ophthalmol* 2011 **89** 111-115.

14. Naeser K, Hjortdal, J.   Multivariate analysis of refractive data: mathematics and statistics of spherocylinders.   *J Cataract Refract Surg* 2001 **27** 129-142.

15. Thibos LN, Wheeler W, Horner D.   Power: an application of Fourier analysis to the description and analysis of refractive error. *Optom Vis Sci* 1997 **74** 367-375.

16. Marsden JE, Tromba AJ.   *Vector Calculus*. 3rd ed. Freeman, New York, 1988 p 256.