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Curves and surfaces in the context of optometry. Part 2: Surfaces

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Abstract

This paper introduces the differential geometry of surfaces in Euclidean 3-space. The first and second fundamental forms of a surface are defined. The first fundamental form provides a metric for calculations of length and area on the surface. The second fundamental form determines surface curvature and, hence, concepts of importance in optometry such as surface power and sagitta. The principal curvatures at a point on a surface are obtained as solutions of a quadratic equation. The torus is used to illustrate the methods.

Much of the life of optometry centres on the concepts of sphere and cylinder; toric surfaces make their appearance here and there. That is for historical reasons. Now modern technology is bringing with it all sorts of new surface shapes. How do we adapt to them? How does our understanding of power hold up when it is based on spheres and cylinders?

A previous paper[1] introduced the differential geometry of space curves and outlined the relevance of concepts to issues of interest in optometry. The goal of this paper is to do the same but for surfaces in space. Clearly surfaces are of paramount importance in optometry; they are the location of much of the process of refraction in spectacle and other lenses and in the eye itself.

Whereas we were able before[1] to outline most of the theory of curves we cannot really hope to do the same for surfaces. The theory of surfaces is a good deal more complicated. All we shall attempt to do here is cover introductory material, essentially that concerning what are known as the first and second fundamental forms of a surface. And we shall do so less formally. Via the first fundamental form we are able to calculate distances and areas in the surface; via the second fundamental form we can determine surface curvature and, hence, quantities of optometric interest such as principal powers and sagitta. The reader who wishes to go further in the study of surfaces should consult the literature, including perhaps the two most popular texts[2, 3] on the subject.

We begin with the vector function of two variables and its derivatives. Differential geometry makes use of a characteristic symbolism which many may find intimidating. However the symbolism greatly reduces the size and number of equations and introduces a transparency without which the subject would be well-nigh impossible. In particular we shall take advantage of Einstein's summation convention. We derive the first fundamental form and show how it is used to calculate lengths and areas in the surface. From the curvature of a curve on the surface we define the concept of curvature of the surface itself and obtain the second fundamental form. From the second fundamental form we obtain the principal curvatures at a point on a surface. We define mean and Gaussian curvature. Principal curvatures lead naturally to a discussion of principal powers. The torus is a surface of considerable importance in astigmatic systems; it is examined in detail in the Appendix as an illustration of the general methods. More particularly the Appendix shows how to calculate the surface area and the principal curvatures at points on the surface of

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the torus. Despite what for many is the menacing appearance of the mathematics the calculations are relatively straightforward.

Vector Function of Two Scalar Variables

Instead of the vector function of one variable that generates a curve (equation 8 of the previous paper[1]) we now have a vector function

$$\mathbf{x} = \mathbf{x}(u,v) \quad (u,v) \in D \tag{1}$$

of two variables that generates a surface. The function is defined over a domain D. One can think of the function as containing three scalar functions of two variables:

$$\mathbf{x} = \begin{pmatrix} x_1(u,v) \\ x_2(u,v) \\ x_3(u,v) \end{pmatrix} \quad (u,v) \in D. \tag{2}$$

Figure 1 is a graphical representation of the surface. The surface is shown by means of two families of curves, the u-parameter curves along which v is constant (one of them is marked v in Figure 1) and the v-parameter curves along which u is constant. In effect the function is a rule for bending and generally distorting a flat patch (the domain D) into some shape in three dimensions. $\mathbf{x}$ is the position vector of a point on the surface. We shall be interested in the surface near point $\mathbf{x}$.

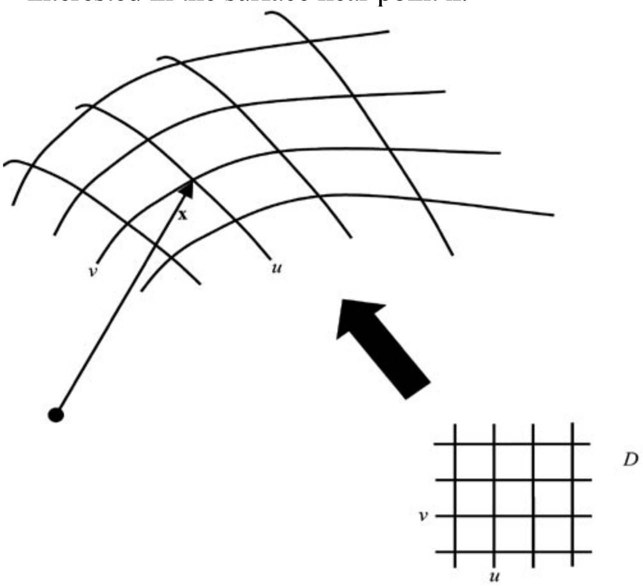

Figure 1 Graphical representation of a vector function of two variables (equation 1). The function generates a surface in space. Along the curve labelled v the parameter v is fixed and parameter u varies; it is a u-parameter curve. Similarly along the curve labelled u parameter u is fixed; it is a v-parameter curve. The surface has two families of curves, the u- and the v-parameter curves. $\mathbf{x}$ is the position vector of the point on the surface with parameters u and v.

Derivatives, Tangents and Notation

What is true of curves in general[1] is true of the u- and v-parameter curves in particular. $\dfrac{\partial \mathbf{x}}{\partial u}$ is a vector tangent to the u-parameter curve at the point with position vector $\mathbf{x}$ and $\dfrac{\partial \mathbf{x}}{\partial v}$ a vector tangent to the v-parameter curve there (Figure 2).

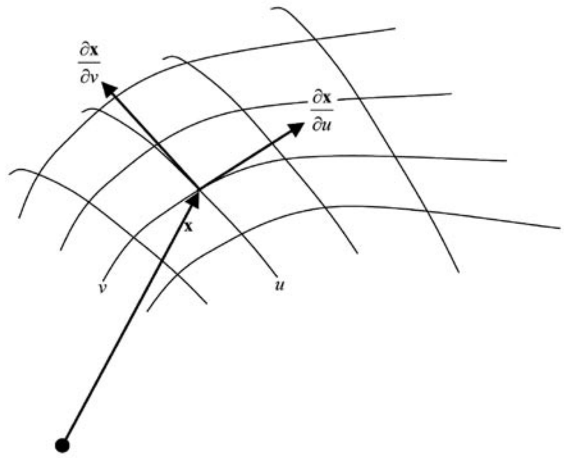

Figure 2 The derivatives $\dfrac{\partial \mathbf{x}}{\partial u}$ and $\dfrac{\partial \mathbf{x}}{\partial v}$ are vectors tangent to the u- and v-parameter curves of the surface at the point with position vector $\mathbf{x}$.

A modified notation allows a more efficient and compact symbolism. Variables u and v are rewritten u^1 and u^2 respectively. Equation 1 then becomes

$$\mathbf{x} = \mathbf{x}(u^1, u^2) \quad (u^1, u^2) \in D \tag{3}$$

and the derivatives become $\dfrac{\partial \mathbf{x}}{\partial u^1}$ and $\dfrac{\partial \mathbf{x}}{\partial u^2}$.

We write u^α to represent either variable u^1 or u^2. The function is abbreviated as

$$\mathbf{x} = \mathbf{x}(u^\alpha) \tag{4}$$

and either derivative by $\dfrac{\partial \mathbf{x}}{\partial u^\alpha}$. The derivatives $\dfrac{\partial \mathbf{x}}{\partial u^\alpha}$ are abbreviated still further as $\mathbf{x}_{,\alpha}$ which means either $\mathbf{x}_{,1}$ or $\mathbf{x}_{,2}$.

The two tangent vectors $\mathbf{x}_{,1}$ and $\mathbf{x}_{,2}$ define the tangent plane of the surface at the point $\mathbf{x}$ (Figure 3). A point in the tangent plane has position vector

$$\mathbf{T} = \mathbf{x} + h\mathbf{x}_{,1} + j\mathbf{x}_{,2} \tag{5}$$

where h and j are real numbers. The vector

$$\mathbf{N} = \mathbf{x}_{,1} \times \mathbf{x}_{,2} \tag{6}$$

is normal to the surface and

$$\hat{\mathbf{N}} = \frac{\mathbf{x}_{,1} \times \mathbf{x}_{,2}}{|\mathbf{x}_{,1} \times \mathbf{x}_{,2}|} \tag{7}$$

is a unit normal vector. If $\mathbf{N} \neq \mathbf{o}$ everywhere the surface is called *regular*.

The differential of $\mathbf{x}$ is

$$d\mathbf{x} = \mathbf{x}_{,1}du^1 + \mathbf{x}_{,2}du^2. \tag{8}$$

One can think of $d\mathbf{x}$ as the displacement in the surface associated with small increments du^1 and du^2 in parameters u^1 and u^2 (Figure 4).

Einstein's Summation Convention

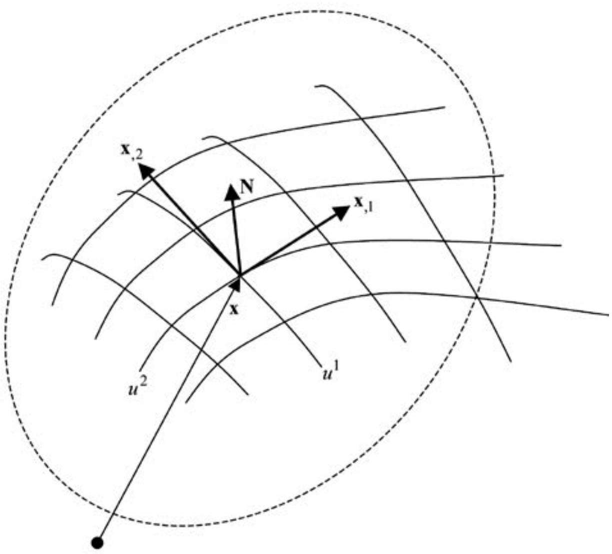

Figure 3 A surface with vectors $\mathbf{x}_{,1}$ and $\mathbf{x}_{,2}$ tangent, at point $\mathbf{x}$, to the parameter curves of the surface. The tangent vectors define the surface's tangent plane (shown by means of a dashed ellipse) and a normal vector $\mathbf{N}$ (equation 6).

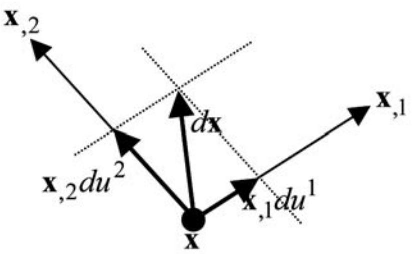

Figure 4 The differential vector $d\mathbf{x}$ as a linear combination (equation 8).

This is a convenient place to introduce what is known as Einsteins's summation convention. Equation 8 can be written as

$$d\mathbf{x} = \sum_{\alpha=1} \mathbf{x}_{,\alpha} du^\alpha. \tag{9}$$

Making use of Einstein's summation convention one writes

$$d\mathbf{x} = \mathbf{x}_{,\alpha} du^\alpha \tag{10}$$

instead. Summation of an expression is understood over a repeated lower-case Greek letter. Changing the letter makes no difference, and so the two ostensibly different expressions $d\mathbf{x} = \mathbf{x}_{,\alpha}du^\alpha$ and $d\mathbf{x} = \mathbf{x}_{,\beta}du^\beta$ are in fact identical. The repeated Greek letter is referred to as a *dummy* index. That is in contrast to an unrepeated Greek letter, as in u^γ for example, which is called a *free* index and over which summation is not understood.

Expressions can contain dummy and free indices and more than one of each. $g^{\gamma\delta}\Gamma_{\delta\alpha\gamma}$, for example, contains two dummy indices, γ and δ, and one free index, α. $g^{\gamma\delta}\Gamma_{\delta\alpha\gamma}$ actually means two things, one for each value of the free index α. In other words $g^{\gamma\delta}\Gamma_{\delta\alpha\gamma}$ implies both $g^{\gamma\delta}\Gamma_{\delta 1\gamma}$ and $g^{\gamma\delta}\Gamma_{\delta 2\gamma}$. Furthermore summation over both γ and δ is understood in both. Thus

$$g^{\gamma\delta}\Gamma_{\delta 1\gamma} = \sum_{\gamma=1}^{2}\sum_{\delta=1}^{2} g^{\gamma\delta}\Gamma_{\delta 1\gamma}$$

or

$$g^{\gamma\delta}\Gamma_{\delta 1\gamma} = g^{11}\Gamma_{111} + g^{12}\Gamma_{211} + g^{21}\Gamma_{112} + g^{22}\Gamma_{212}. \tag{11}$$

Similarly

$$g^{\gamma\delta}\Gamma_{\delta 2\gamma} = g^{11}\Gamma_{121} + g^{12}\Gamma_{221} + g^{21}\Gamma_{122} + g^{22}\Gamma_{222}. \tag{12}$$

Thus $g^{\gamma\delta}\Gamma_{\delta\alpha\gamma}$ is a compact way of writing the right-hand sides of both equations 11 and 12 simultaneously. (We shall meet g below. Γ refers to what are called the Christoffel symbols, objects that are beyond the scope of this paper.)

The First Fundamental Form of a Surface

The squared length of the differential vector $d\mathbf{x}$ is called the *first fundamental form* of the surface. It is given by

$$(ds)^2 = d\mathbf{x} \cdot d\mathbf{x}. \tag{13}$$

Substituting from equation 8 we obtain

$$(ds)^2 = (\mathbf{x}_{,1}du^1 + \mathbf{x}_{,2}du^2) \bullet (\mathbf{x}_{,1}du^1 + \mathbf{x}_{,2}du^2)$$

or

$$(ds)^2 = \mathbf{x}_{,1} \bullet \mathbf{x}_{,1}(du^1)^2 + 2\mathbf{x}_{,1} \bullet \mathbf{x}_{,2}du^1du^2 + \mathbf{x}_{,2} \bullet \mathbf{x}_{,2}(du^2)^2. \quad (14)$$

We define

$$g_{\alpha\beta} = \mathbf{x}_{,\alpha} \bullet \mathbf{x}_{,\beta} \quad (15)$$

and then rewrite equation 14 as

$$(ds)^2 = g_{11}(du^1)^2 + 2g_{12}du^1du^2 + g_{22}(du^2)^2. \quad (16)$$

In terms of matrices this can be written as

$$(ds)^2 = \begin{pmatrix} du^1 & du^2 \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} du^1 \\ du^2 \end{pmatrix} \quad (17)$$

in which

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \quad (18)$$

is a symmetric matrix. The $g_{\alpha\beta}$ are called the *coefficients* of the first fundamental form of the surface and g the *matrix* of the first fundamental form of the surface. One can make use of Einstein's summation convention and write the first fundamental form (equation 16) as

$$(ds)^2 = g_{\alpha\beta}du^\alpha du^\beta. \quad (19)$$

The matrix of the first fundamental form of a torus is calculated in the Appendix.

Length of a Curve in a Surface

Consider now a surface defined by $\mathbf{x} = \mathbf{x}(u^1, u^2)$. Suppose we make u^1 and u^2 both functions of a variable t, that is,

$$u^1 = u^1(t) \text{ and } u^2 = u^2(t), \quad (20)$$

on an interval I. We then have

$$\mathbf{x} = \mathbf{x}\big(u^1(t), u^2(t)\big)$$

which is a new vector function but now of a single variable (t):

$$\mathbf{x} = \mathbf{x}(t). \quad (21)$$

It represents a curve on the surface (Figure 5).

The first fundamental form of a surface allows us to calculate the lengths of curves on it. The length is given by integrating elements of length $|ds|$, that is, $\sqrt{g_{\alpha\beta}du^\alpha du^\beta}$ (equation 19), along the curve for an appropriate interval of the variable t. The length of the curve is given by

$$S = \int_{t_1}^{t_2} \sqrt{g_{\alpha\beta}\frac{du^\alpha}{dt}\frac{du^\beta}{dt}}\, dt.$$

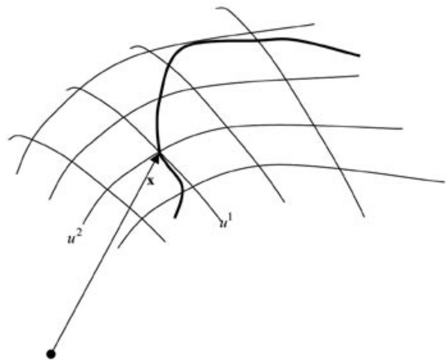

Figure 5 A curve (equation 21) on a surface.

Area of a Surface

Figure 6 shows a closed curve on a surface. The first fundamental form allows one to calculate the area of the surface enclosed by the curve. The formula is

$$A = \iint_\Sigma \sqrt{\det g}\, du^1du^2 \quad (23)$$

where g is the matrix of the first fundamental form (equation 18).

The surface area of a torus is calculated in the Appendix.

Normal Curvature of a Curve on a Surface

At point $\mathbf{x}$ a curve has curvature vector $\ddot{\mathbf{x}}$ (Figure 7). We can write the curvature vector as a linear combination of the three vectors $\hat{\mathbf{N}}$, $\mathbf{x}_{,1}$ and $\mathbf{x}_{,2}$:

$$\ddot{\mathbf{x}} = k\hat{\mathbf{N}} + \lambda\mathbf{x}_{,1} + \mu\mathbf{x}_{,2.}$$

It follows that

$$k = \ddot{\mathbf{x}} \bullet \hat{\mathbf{N}} \qquad (24)$$

which is what we call the *normal curvature* of the curve at $\mathbf{x}$ in the surface.

Curvature of a Surface

We obtain an expression for the curvature vector $\ddot{\mathbf{x}}$ from equation 8. The first derivative with respect to s is

$$\dot{\mathbf{x}} = \mathbf{x}_{,1}\dot{u}^1 + \mathbf{x}_{,2}\dot{u}^2.$$

Differentiation gives the second derivative:

$$\ddot{\mathbf{x}} = \mathbf{x}_{,1}\ddot{u}^1 + \dot{\mathbf{x}}_{,1}\dot{u}^1 + \mathbf{x}_{,2}\ddot{u}^2 + \dot{\mathbf{x}}_{,2}\dot{u}^2$$

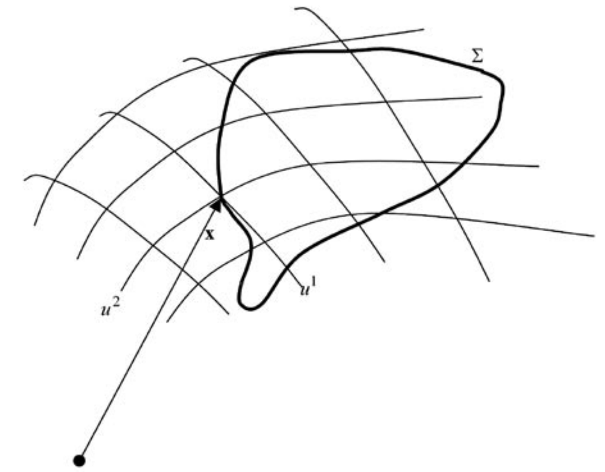

Figure 6 A closed curve on a surface. The portion of the surface enclosed by the curve is denoted Σ. The area of that portion is given by equation 23.

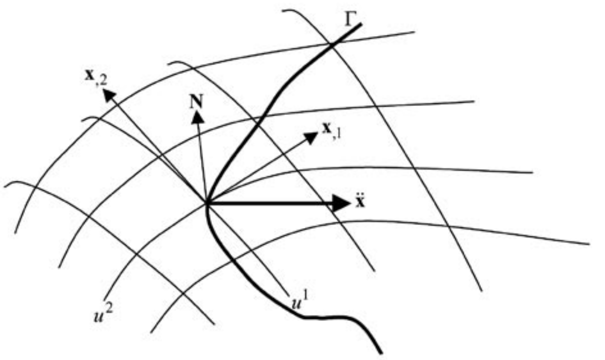

Figure 7 The curvature vector $\ddot{\mathbf{x}}$ at the point $\mathbf{x}$ of the curve Γ. The normal curvature of the surface at $\mathbf{x}$ in the direction of Γ is given by equation 24.

where by $\dot{\mathbf{x}}_{,1}$ we mean $\frac{d}{ds}\frac{\partial \mathbf{x}}{\partial u^1}$. Substituting into equation 24 gives

$$k = \dot{\mathbf{x}}_{,1} \bullet \hat{\mathbf{N}}\,\dot{u}^1 + \dot{\mathbf{x}}_{,2} \bullet \hat{\mathbf{N}}\,\dot{u}^2. \qquad (25)$$

Because $\mathbf{x} = \mathbf{x}(u^1, u^2)$ it follows that

$$\mathbf{x}_{,1} = \mathbf{x}_{,1}(u^1, u^2)$$

whose differential is

$$d\mathbf{x}_{,1} = \frac{\partial \mathbf{x}_{,1}}{\partial u^1} du^1 + \frac{\partial \mathbf{x}_{,1}}{\partial u^2} du^2$$

or

$$d\mathbf{x}_{,1} = \mathbf{x}_{,11} du^1 + \mathbf{x}_{,12} du^2.$$

Hence

$$\dot{\mathbf{x}}_{,1} = \mathbf{x}_{,11}\dot{u}^1 + \mathbf{x}_{,12}\dot{u}^2$$

and similarly for $\dot{\mathbf{x}}_2$. Together the two equations can be written as one:

$$\dot{\mathbf{x}}_{,\alpha} = \mathbf{x}_{,\alpha\beta}\dot{u}^\beta.$$

Substituting into equation 25 we obtain

$$k = \mathbf{x}_{,11} \bullet \hat{\mathbf{N}}(\dot{u}^1)^2 + 2\mathbf{x}_{,12} \bullet \hat{\mathbf{N}}\dot{u}^1\dot{u}^2 + \mathbf{x}_{,22} \bullet \hat{\mathbf{N}}(\dot{u}^2)^2$$

where we have made use of the fact that $\mathbf{x}_{,12} = \mathbf{x}_{,21}$. We write this equation as

$$k = b_{11}(\dot{u}^1)^2 + 2b_{12}\dot{u}^1\dot{u}^2 + b_{22}(\dot{u}^2)^2 \qquad (26)$$

where

$$b_{\alpha\beta} = \mathbf{x}_{,\alpha\beta} \bullet \hat{\mathbf{N}}. \qquad (27)$$

Using the summation convention we can write equation 26 as

$$k = b_{\alpha\beta}\dot{u}^\alpha\dot{u}^\beta$$

or, in terms of differentials, the normal curvature of the curve in the surface through point $\mathbf{x}$ is

$$k = b_{\alpha\beta}du^\alpha du^\beta/(ds)^2. \qquad (28)$$

Coefficients $b_{\alpha\beta}$ are properties of the surface. The differentials in equation 28 represent the direction of the curve in the surface at **x**. It follows that all curves in the surface that have the same direction at **x** have the same normal curvature. It also follows that k can be regarded as the curvature of the surface in the direction defined by the differentials.

The Second Fundamental Form of a Surface

The expression

$$k(ds)^2 = b_{\alpha\beta}du^\alpha du^\beta \tag{29}$$

is called the *second fundamental form* of the surface. The matrix

$$b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{30}$$

is the matrix of the second fundamental form of the surface. It is calculated for a torus in the Appendix.

Surface Curvature and Power

Matrix b defines the shape of the surface in the neighbourhood of the point **x**. In fact one may think of it as a curvature matrix for the surface. If b is null then the surface there is planar. Otherwise the surface is curved at the point. Points on a surface are classified into types depending on the determinant of b. If $\det b > 0$ then **x** is called an *elliptic* point; if $\det b = 0$ (but $b \neq \mathbf{O}$) then it is a parabolic point; if $\det b < 0$ then it is a hyperbolic point. In optometric terms the surface is described as *mixed* in the neighbourhood of point **x** if **x** is hyperbolic, *unmixed* if **x** is *elliptic* and *cylindrical* if **x** is parabolic. A torus has points of all three types. One can think of the inner tube for the wheel of a car. The elliptic points are those on the tube that would touch the ground when it rolls; the parabolic points are those that would touch the ground when the tube lies sideways; the hyperbolic points are those that are adjacent to the rim of the wheel.

If the surface separates two transparent media then the dioptric power of the surface in the neighbourhood of point **x** takes the form

$$\mathbf{F} = b\Delta n$$

where **F** is the dioptric power matrix. Δn is the difference in the indices of refraction across the surface taken in the appropriate order.

Sagitta

It turns out that, for points near **x**, the second fundamental form (equation 29) is twice the perpendicular distance between the surface and the tangent plane (Figure 3). In optometric terms this is called the sagitta of the surface. More specifically we can write the sagitta as

$$z = \tfrac{1}{2} b_{\alpha\beta}du^\alpha du^\beta. \tag{31}$$

This would be an approximate expression for the sagitta. An exact expression for sagitta is given by

$$z = (\mathbf{x} - \mathbf{T}) \bullet \hat{\mathbf{N}}.$$

Principal Curvatures and Principal Powers

Equation 28 gives the curvature of the surface at **x** in the direction defined by the differentials. The curvature k usually depends on the direction, being a maximum k_+ in one direction and a minimum k_- in another. k_+ and k_- are called the *principal curvatures* of the surface at **x** and the directions in the surface at **x** are the corresponding *principal directions*.

Finding the principal curvatures and directions is an eigenvalue problem. We give no details here except to say that the principal curvatures turn out to be the solutions to the quadratic equation

$$k^2 - k\,\frac{g_{11}b_{22} + g_{22}b_{11} - 2g_{12}b_{12}}{g_{11}g_{22} - g_{12}^2} + \frac{b_{11}b_{22} - b_{12}^2}{g_{11}g_{22} - g_{12}^2} = 0 \tag{32}$$

The principal curvatures are calculated for a torus in the Appendix.

The *mean curvature* is defined as

$$\mu = \tfrac{1}{2}(k_+ + k_-) \tag{33}$$

and the *Gaussian curvature*

$$K = k_+k_- . \tag{34}$$

The principal powers are related to the principal curvatures by familiar expressions of the form

$F_{\pm} = k_{\pm}\Delta n.$

From equation 33 we see that the nearest equivalent sphere is related to the mean curvature by

$F_{\mathrm{I}} = \mu\Delta n.$

Concluding Remarks

This is as far as we shall go in our study of curves and surfaces. The reader who would like to go further can turn to the literature[2, 3].

As we have seen curves are generated by vector functions of a single variable; they have curvature and torsion that in general vary with position along them. Surfaces are generated by vector functions of two variables. Distance and area in the surface can be determined by the first fundamental form of the surface; the shape of a surface, including its curvature in particular, is determined by the second fundamental form.

Spherical and cylindrical surfaces, and the optometric concepts that depend on them, are going increasingly to be out of place in a modern optometry of aspheric, ellipsoidal and varifocal surfaces. Our educational programs, in particular, are still founded in spherocylindrical terms and do not really adequately prepare students for that modern optometry. Differential geometry is the key to an understanding of surface shape. Typically, though, it is presented at a third-year level in applied mathematics courses at university which makes it unrealistic to expect it to be given a slot in the curriculum in the foreseeable future. But there is a need, I believe, for some optometric academics to gain expertise in the subject and explore ways of channelling its concepts into the profession.

Acknowledgement

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Appendix

Consider the parametrization

$$\mathbf{x} = \begin{pmatrix} (q + p\sin\phi)\cos\theta \\ (q + p\sin\phi)\sin\theta \\ p\cos\phi \end{pmatrix} \quad \theta \in [0, 2\pi),\ \ \phi \in [0, 2\pi)$$

with $q > p > 0$. We illustrate the methods by determining the matrix of the first fundamental form and, hence, the surface area of the surface. We then determine the matrix of the second fundamental form and, hence, the principal curvatures.

The reader is encouraged first to check that the surface is that of a torus, to interpret the fixed quantities p and q and the variable quantities θ and ϕ, and to work out how the torus is lying relative to the set of orthogonal axes. This can be done by first choosing two fixed numbers for p and q (say 1 and 2) and then working out $\mathbf{x}$ for a few key values of θ and ϕ in the domain.

The two first derivatives are

$$\mathbf{x}_{,\theta} = \begin{pmatrix} -(q + p\sin\phi)\cos\theta \\ (q + p\sin\phi)\cos\theta \\ 0 \end{pmatrix}$$

and

$$\mathbf{x}_{,\phi} = \begin{pmatrix} p\cos\phi\cos\theta \\ p\cos\phi\sin\theta \\ -p\sin\phi \end{pmatrix}.$$

The coefficients of the first fundamental form are then obtained from equation 15. For example

$$g_{\theta\theta} = \mathbf{x}_{,\theta} \bullet \mathbf{x}_{,\theta} = (q + p\sin\phi)^2.$$

The result is the matrix of the first fundamental form

$$g = \begin{pmatrix} (q + p\sin\phi)^2 & 0 \\ 0 & p^2 \end{pmatrix}.$$

Hence

$\det g = p^2(q + p\sin\phi)^2$.

The area A of the torus is given by equation 22:

$$A = \int_0^{2\pi}\int_0^{2\pi} p(q + \sin\phi)\,d\theta\,d\phi.$$

$$= \int_0^{2\pi} (q + \sin\phi)2\pi\,d\phi$$

$$= 4\pi^2 pq.$$

For the matrix of the second fundamental form we first need a normal vector (equation 6):

$$\mathbf{N} = \mathbf{x}_{,\theta} \times \mathbf{x}_{,\phi}$$

$$= \begin{pmatrix} -(q + p\sin\phi)\sin\theta & p\cos\phi + \cos\theta & \mathbf{i} \\ q + p\sin\phi)\sin\theta & p\cos\phi + \cos\theta & \mathbf{j} \\ 0 & -\mathrm{p}\sin\phi & \mathbf{k} \end{pmatrix}$$

$$= -p(q + \sin\phi)\begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}.$$

Its magnitude is

$|\mathbf{N}| = p(q + \sin\phi)$.

Hence

$$\hat{\mathbf{N}} = -\begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix}$$

(equation 7) is a unit normal. The second derivatives are

$$\mathbf{x}_{,\theta\theta} = \begin{pmatrix} -(q + p\sin\phi)\cos\theta \\ -(q + p\sin\phi)\sin\theta \\ 0 \end{pmatrix},$$

$$\mathbf{x}_{,\theta\phi} = \begin{pmatrix} -p\cos\phi\sin\theta \\ p\cos\phi\sin\theta \\ 0 \end{pmatrix}$$

and

$$\mathbf{x}_{,\phi\phi} = \begin{pmatrix} -p\sin\phi\cos\theta \\ -p\sin\phi\sin\theta \\ 0 \end{pmatrix}.$$

The coefficients of the second fundamental form of the surface are obtained using equation 27. For example $b_{\theta\theta} = \mathrm{x}_{,\theta\theta} \bullet \hat{\mathbf{N}}$ Equation 30 gives the matrix of the second fundamental form of the torus:

$$b = \begin{pmatrix} (q + p\sin\phi)\sin\phi & 0 \\ 0 & p \end{pmatrix}.$$

From equation 33 we obtain a quadratic equation for the principal curvatures

$$k^2 - k\,\frac{q + 2p\sin\phi}{p(q + p\sin\phi)} + \frac{\sin\phi}{p(q + p\sin\phi)} = 0.$$

The solutions turn out to be

$$k_\pm = \frac{q + 2p\sin\phi \pm q}{2p(q + p\sin\phi)},$$

that is, the principal curvatures at a point on the surface are

$$k_+ = \frac{1}{p} \text{ and } k_- = \frac{\sin\phi}{q + p\sin\phi}.$$

Notice that k_+ is constant over the surface while k_- is positive, zero or negative according to the value of the parameter ϕ .