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Quantitative analysis in Hamiltonian space of the transformed ray transferences of a cornea*

SD Mathebula[†], A Rubin[‡] and WF Harris[††]

[†]Department of Optometry, University of Limpopo, Private Bag X1106, Sovenga, 0727 South Africa, and [‡,††]Optometric Science Research Group, Department of Optometry, University of Johannesburg, PO Box 524, Auckland Park, 2006 South Africa

[†]<solanim@ul.ac.za>
[‡]<arubin@uj.ac.za>
[††]<wharris@uj.ac.za>

Abstract

The primary purpose of this paper is to illustrate the quantitative analysis of the linear-optical character of a cornea using transformed ray transferences in a 10-dimensional Hamiltonian linear space. A Pentacam was utilized to obtain 43 successive measurements of the powers of the anterior and posterior corneal surfaces and the central corneal thicknesses of the right eye of a single subject. From these measurements 4×4 ray transferences were calculated (and principal matrix logarithms for all the transferences were determined). This produced a set of 43 transformed transferences for the cornea which represent 43 points in a 10- dimensional Hamiltonian space. A 10-component mean and 10×10 variance-covariance matrix were calculated from the transformed transferences. The mean, and variances and covariances represent, in the 10-space, the average and the spread respectively of the measurements characterising the optical nature of the cornea. The matrix exponential of the mean gives a value for the mean transference of the cornea; it represents the average cornea. The analysis described here can be applied to most optical systems including whole eyes and is complete within linear optics. We believe it to be the first such analysis.

Key words: dioptric power, Hamiltonian matrices, Hamiltonian space, linear optics, ray transference, symplecticity, Scheimpflug photography, Pentacam

Introduction

It is now widely recognised in the research environment that the sphero-cylindrical representation of power is not suitable for quantitative analysis.[1-5] Quantitative analysis of powers require the conversion of spherocylindrical powers to dioptric power matrices (or related entities) which then can be treated by conventional mathematical and statistical methods. One finds an average[3] of *N* refractive states, for example, by adding them as dioptric power matrices and dividing by *N*. However there is much more to the optics of an eye than just its refractive state. Eyes with the same refractive state do not in general have the same retinal image sizes, shapes, or orientations for example[6]. The question then is: How can one perform complete quantitative analyses on the optical character of whole eyes and of optical systems in general?

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[†]BOptom MOptom (UNIN)
[‡]DipOptom CAS(NewEnCo) MPhil (FRSSAF) DPhil (RAU)
[††]Bsc(Eng) MS PhD(Minnesota) BOptom(RAU) HonsBsc(Unisa)

The answer lies in the ray transference. It represents a complete quantification of the first-order optical character of the system. The basic theory is presented in two accompanying papers[7, 8]. The purpose of this paper is show the application of the theory to basic calculations made on the cornea of a human eye with explicit allowance for thickness. By choosing this system we are taking more or less the simplest system that exhibits the features of thick systems and yet, in some sense, is not far from being a thin system. From a set of measurements made on a corneal computer program (in Matlab) calculates an average cornea and a measure of the spread or variation of the data.

Mathematical background

The dioptric power matrix can be expanded as

$$\mathbf{F} = F_{\mathrm{I}}\mathbf{I} + F_{\mathrm{J}}\mathbf{J} + F_{\mathrm{K}}\mathbf{K} + F_{\mathrm{L}}\mathbf{L} \tag{1}$$

in general. (For details of the mathematics the reader is referred elsewhere[7-10] and to the papers cited therein.) For thin systems the matrix is symmetric: the last term vanishes and one is left with

$$\mathbf{F} = F_{\mathrm{I}}\mathbf{I} + F_{\mathrm{J}}\mathbf{J} + F_{\mathrm{K}}\mathbf{K}\,. \tag{2}$$

The power can be written as a coordinate vector:

$$\mathbf{f} = \begin{pmatrix} F_{\mathrm{I}} & F_{\mathrm{J}} & F_{\mathrm{K}} \end{pmatrix}'\,. \tag{3}$$

Suppose one has a sample of N powers. A mean can be calculated as an arithmetic mean of the dioptric power matrices or of the coordinate vectors. The variance-covariance based on **f** is a 3×3 symmetric matrix of the form

$$\mathbf{S} = \begin{pmatrix} s_{\mathrm{II}} & s_{\mathrm{IJ}} & s_{\mathrm{IK}} \\ s_{\mathrm{JI}} & s_{\mathrm{JJ}} & s_{\mathrm{JK}} \\ s_{\mathrm{KI}} & s_{\mathrm{KJ}} & s_{\mathrm{KK}} \end{pmatrix}. \tag{4}$$

In the most general case the ray transference is a real 5×5 matrix[6, 7, 11]. If one is not interested in prismatic effects and the effects of decentration of elements in the system then the transference can be reduced to a 4×4 matrix which takes the form

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}. \tag{5}$$

The dioptric power matrix **F** is simply related to **C** by[1]

$$\mathbf{F} = -\mathbf{C}\,. \tag{6}$$

The transference **T** of a system can be calculated by multiplying the transferences of subsystems in reverse order[1, 11]. For a cornea in particular the transference is given by

$$\mathbf{T} = \mathbf{T}_{\mathrm{p}}\mathbf{T}_{\mathrm{c}}\mathbf{T}_{\mathrm{a}} \tag{7}$$

where $\mathbf{T}_{\mathrm{a}}$ is the transference of the anterior surface of the cornea, $\mathbf{T}_{\mathrm{p}}$ that of the posterior surface and $\mathbf{T}_{\mathrm{c}}$ is the transference of the body of the cornea. $\mathbf{T}_{\mathrm{a}}$ is given by

$$\mathbf{T}_{\mathrm{a}} = \begin{pmatrix} \mathbf{I} & \mathbf{O} \\ -\mathbf{F}_{\mathrm{a}} & \mathbf{I} \end{pmatrix} \tag{8}$$

where $\mathbf{F}_{\mathrm{a}}$ is the power of the anterior surface. **I** is the 2×2 identity matrix and **O** the 2×2 null matrix. $\mathbf{T}_{\mathrm{p}}$ is given by equation 8 with subscript a replaced by p. $\mathbf{T}_{\mathrm{c}}$ is given by

$$\mathbf{T}_{\mathrm{c}} = \begin{pmatrix} \mathbf{I} & \tau\mathbf{I} \\ \mathbf{O} & \mathbf{I} \end{pmatrix} \tag{9}$$

where τ is the reduced thickness of the cornea.

Because of symplecticity, transferences themselves are not closed under addition and multiplication by a scalar and, hence, are not amenable to conventional quantitative analysis. Transferences must first be transformed into Hamiltonian matrices[7, 8, 12]. Once transformed they constitute a vector (or linear) space and standard quantitative analysis such as the calculation of means and variance-covariances can be performed in what can be called transformed transference space.[8] An accompanying paper[7] describes several ways of transforming transferences to Hamiltonian matrices. The one adopted here is the principal matrix logarithm. Thus for each transference **T** we define the transformed transference by

$$\hat{\mathbf{T}} = \operatorname{Log}\mathbf{T}\,. \tag{10}$$

The transformed transference can be written in terms of sub-matrices

$$\hat{\mathbf{T}} = \begin{pmatrix} \hat{\mathbf{A}} & \hat{\mathbf{B}} \\ \hat{\mathbf{C}} & \hat{\mathbf{D}} \end{pmatrix} \tag{11}$$

just as for the transference itself (equation 5). Quantitative analysis can be performed on the transformed transferences. The Hamiltonian space in which the calculations are performed is 10-dimensional.

$\hat{\mathbf{A}}$ in equation 11 can be expanded exactly as in equation 1:

$$\hat{\mathbf{A}} = \hat{A}_{\mathrm{I}}\mathbf{I} + \hat{A}_{\mathrm{J}}\mathbf{J} + \hat{A}_{\mathrm{K}}\mathbf{K} + \hat{A}_{\mathrm{L}}\mathbf{L} \tag{12}$$

Because of the Hamiltonian nature of $\hat{\mathbf{T}}$ both $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ are symmetric; they can be expanded as in equation 2. The four coefficients of $\hat{\mathbf{A}}$ and the three each of $\hat{\mathbf{B}}$ and $\hat{\mathbf{C}}$ are combined together to make up the 10-component coordinate vector

$$\hat{\mathbf{v}} = \begin{pmatrix} \hat{A}_{\mathrm{I}} & \hat{A}_{\mathrm{J}} & \hat{A}_{\mathrm{K}} & \hat{A}_{\mathrm{L}} & \hat{B}_{\mathrm{I}} & \hat{B}_{\mathrm{J}} & \hat{B}_{\mathrm{K}} & \hat{C}_{\mathrm{I}} & \hat{C}_{\mathrm{J}} & \hat{C}_{\mathrm{K}} \end{pmatrix}'. \tag{13}$$

(Equation 20 in an accompanying paper[8] gives the same coordinate vector except that it includes the four additional entries necessary to account for prismatic effects in the system.)

From a set of N values of coordinate vector $\hat{\mathbf{v}}$ one can obtain a mean and a variance-covariance matrix as defined by equations 22 and 23 of an accompanying paper[8]. The variance-covariance matrix is a 10×10 symmetric matrix of 10 variances and 45 covariances.

Where desired, transformed transferences can be converted back to (untransformed) transferences by means of the matrix exponential:

$$\tilde{\mathbf{T}} = \exp\hat{\mathbf{T}}\,. \tag{14}$$

To determine the transference of a cornea one needs measurements of the three elements of the cornea, namely the powers of the axial anterior and posterior corneal surfaces, and also the axial or central corneal thickness. The refractive indices of the cornea and the aqueous humour were taken as 1.376 and 1.336 respectively. The Pentacam (70700, Oculus, Wetzlar, Germany) provides some of the aforementioned information. It is a novel instrument which uses a rotating Scheimpflug camera and a monochromatic slit light source (LED emitting blue light of wavelength 475 nm) that rotate together around the optical axis of the eye to image the anterior segment of the eye.[13-16] Within about two seconds the camera rotates 180° and 12, 25 or 50, meridional or cross-sectional slit images (depending on the user setting) are measured. Each image contains 500 measurement points on the front and back corneal surfaces from which a true elevation map is constructed. In total up to 25 000 measurements of height were processed to produce a three-dimensional model of the anterior segment of the eye. This model provided data on anterior chamber biometry (depth, angle and volume), corneal topography (anterior and posterior corneal surfaces) and simulated keratometry.

Method

Ten students studying optometry at the University of Johannesburg volunteered as subjects for the study. This paper analyzes data from the right eye of just one of them, Subject 1. The study complied with the necessary ethical requirements of the Ethics Committee of the University of Johannesburg. Volunteers were informed about the purpose of the study and had to give informed consent before inclusion. All subjects had normal eyes without ocular abnormalities as verified by biomicroscopy, ocular pupillary function and direct ophthalmoscopy. All subjects had to have a visual acuity of 6/6 or better with subjective refraction compensated. Subjective refraction was determined using a phoropter with target at a distance of 6 m. All subjects had to be healthy, not wearing contact lenses, no history of corneal trauma or surgery and not taking medication that might interfere with normal visual function. Some of this information was obtained via a subject questionnaire. The primary author (SDM) took all measurements. About 43 successive measurements were taken for the right eye of each subject using the Pentacam (43 measurements were made in order to allow for possible removal of outliers but with final retention of at least 40 measurements per eye.)

The Pentacam system consists of the Pentacam itself and a personal computer. The software is fully automatic. After the subject's personal data (name and date of birth) were entered, the program was changed to imaging mode. Each subject sat in front of the camera with the chin on the chin rest and the forehead against the forehead strap and looked straight at a fixation target in the centre of the field. The real-time image of the subject's eye appeared on the computer screen. With the machine marking the pupil edge and center, and the corneal apex, the image was focused, aligned and centred manually by moving the Pentacam. The Pentacam's automatic release mode was used. In this mode the instrument automatically determines when correct focus and alignment with the corneal apex have been achieved and then performs a scan. In less than two seconds the rotating camera captured 25 slit images for each of the 43 successive measurements of the anterior segment of the eye. The Pentacam was refocused prior to each measurement set. Subjects were allowed to remove their heads from the instrument while it was processing the Scheimpflug images. It took about one hour to obtain the necessary Pentacam measurements for Subject 1. Each measurement took about a minute with focusing and alignment. After the measurement session all 43 Scheimpflug maps were printed for the subject and from these maps 43 measurements of the anterior and posterior corneal keratometric measurements along principal meridians and the axial corneal thicknesses were recorded.

The forty-three simulated keratometry measurements were transformed into dioptric power matrices for the anterior surface of the cornea. The mean power was calculated and expressed in the form of equation 2. The 3×3 variance-covariance matrix was calculated based on this form. The same was done for the posterior corneal surface.

Transferences were calculated for each of the three elements of the cornea: the anterior surface of the cornea (equation 8), the body of the cornea (equation 9) and the posterior surface of the cornea (equation 8 but with subscript p instead of a). The transference $\mathbf{T}$ of the cornea was calculated according to equation 7. This gave 43 transferences $\mathbf{T}_1$, $\mathbf{T}_2$, …, $\mathbf{T}_{43}$. Forty-three transformed transferences $\hat{\mathbf{T}}_1$, $\hat{\mathbf{T}}_2$, …, $\hat{\mathbf{T}}_{43}$ were calculated according to equation 10. The transformed transferences were expressed as coordinate vectors (equation 13) $\hat{\mathbf{v}}_1$, $\hat{\mathbf{v}}_2$, …, $\hat{\mathbf{v}}_{43}$. A mean $\tilde{\hat{\mathbf{v}}}$ was calculated according to equation 22 of an accompanying paper[8] and a variance-covariance $\hat{\mathbf{S}}$ according to equation 23 of that paper. From $\tilde{\hat{\mathbf{v}}}$ the mean log-transference $\tilde{\hat{\mathbf{T}}}$ of the cornea was calculated. Finally, from $\tilde{\hat{\mathbf{T}}}$ the average transference of the cornea $\tilde{\mathbf{T}}$ was calculated by means of equation 14.

Results

Figure 1 shows stereo-pair scatter plots representing the dioptric power of the anterior (part **a**) and posterior (part **b**) surfaces of the cornea of Subject 1. The origin is at the sample mean in each case. Each dot represents one of the 43 measurements. Various statistics are presented in Table 1 and the mean power is presented both in conventional spherocylindrical form and as a dioptic power matrix.

The calculated transferences of the cornea are listed in Table 2 (but to simplify matters and save space only the first two and last one of the measurements are shown) and the corresponding coordinate vectors of the log-transferences are listed in Table 3 (again, only a few results are shown). But, Figure 2 shows the variation of the components of the coordinate vectors across all 43 measurements. The mean of the coordinate vectors of the log-transferences is given in Table 4; the mean log-transference itself is also provided. Table 5 shows the variance-covariance matrix of the log-transferences of the cornea, and lastly, Table 6 presents the average transference of the cornea recovered from the mean log-transference.

Table 1. Statistics for the anterior and posterior corneal surfaces of Subject 1: the mean powers, the variance and covariances based on the coordinate vector $\begin{pmatrix} F_{\mathrm{I}} & F_{\mathrm{J}} & F_{\mathrm{K}} \end{pmatrix}'$ and the volumes of the ellipsoids of constant probability density containing an estimated 95% of the population of measurements.

	Anterior corneal surface	**Posterior corneal surface**
Mean corneal power $F_{\mathrm{S}}\ F_{\mathrm{C}} \times A$ $\mathbf{F}$	$43.19\ -0.44 \times 86$ $42.97\mathbf{I} - 0.22\mathbf{J} + 0.03\mathbf{K}$ D	$-5.91\ -0.37\ 175$ $-6.09\mathbf{I} + 0.18\mathbf{J} - 0.03\mathbf{K}$ D
Variances and covariances (**S**)	$\begin{pmatrix} 0.0166 & -0.0065 & 0.0003 \\ -0.0065 & 0.0068 & 0.0006 \\ 0.0003 & 0.0006 & 0.0090 \end{pmatrix}\mathrm{D}^2$	$\begin{pmatrix} 0.0053 & 0.0031 & -0.006 \\ 0.0031 & 0.0044 & -0.0002 \\ -0.0006 & -0.0002 & 0.0008 \end{pmatrix}\mathrm{D}^2$
Volume of ellipsoid	0.20 D^3	0.03 D^3

Table 2. The first two and the last one of the 43 4×4 transferences determined for Subject 1.

$$\mathbf{T}_1 = \begin{pmatrix} 0.9786 & 0.0000 & 0.0005\ \mathrm{m} & 0\ \mathrm{m} \\ 0.0000 & 0.9783 & 0\ \mathrm{m} & 0.0005\ \mathrm{m} \\ -41.8128\ \mathrm{D} & 0.0106\ \mathrm{D} & 1.0026 & -0.0000 \\ 0.0106\ \mathrm{D} & -41.9378\ \mathrm{D} & -0.0000 & 1.0028 \end{pmatrix}$$

$$\mathbf{T}_2 = \begin{pmatrix} 0.9780 & 0.0000 & 0.0005\ \mathrm{m} & 0\ \mathrm{m} \\ 0.0000 & 0.9779 & 0\ \mathrm{m} & 0.0005\ \mathrm{m} \\ -41.9551\ \mathrm{D} & 0.0986\ \mathrm{D} & 1.0027 & 0.0000 \\ 0.0986\ \mathrm{D} & -41.9317\ \mathrm{D} & 0.0000 & 1.0029 \end{pmatrix}$$

$$\vdots$$

$$\mathbf{T}_{43} = \begin{pmatrix} 0.9780 & -0.0000 & 0.0005\ \mathrm{m} & 0\ \mathrm{m} \\ 0.0000 & 0.9783 & 0\ \mathrm{m} & 0.0005\ \mathrm{m} \\ -42.6050\ \mathrm{D} & -0.0088\ \mathrm{D} & 1.0024 & 0.0000 \\ -0.0088\ \mathrm{D} & -42.1952\ \mathrm{D} & 0.0000 & 1.0029 \end{pmatrix}$$

Table 3. The first two and the last of the 43 10-component Hamiltonian coordinate vectors for the transformed transference of the cornea. Figure 3 shows plots of the 10 components from all 43 transferences.

$$\hat{\mathbf{v}}_1 = \begin{pmatrix} -.0122 & .0001 & .0000 & -.0000 & -.0005\ \mathrm{m} & -.0000\ \mathrm{m} & -.0000\ \mathrm{m} & -42.0070\ \mathrm{D} & 0.0629\ \mathrm{D} & -0.0000\ \mathrm{D} \end{pmatrix}'$$

$$\hat{\mathbf{v}}_2 = \begin{pmatrix} -.0122 & .0001 & .0000 & -.0000 & -.0005\ \mathrm{m} & .0000\ \mathrm{m} & -.0000\ \mathrm{m} & -42.0783\ \mathrm{D} & -0.0118\ \mathrm{D} & -0.0000\ \mathrm{D} \end{pmatrix}'$$

$$\vdots$$

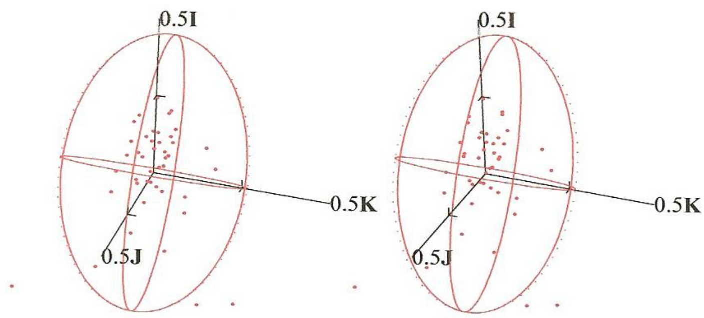
$$\hat{\mathbf{v}}_{43} = \begin{pmatrix} -.0124 & .0002 & -.0000 & .0000 & .0005\ \mathrm{m} & .0000\ \mathrm{m} & .0000\ \mathrm{m} & -42.5385\ \mathrm{D} & -0.2063\ \mathrm{D} & 0.0000\ \mathrm{D} \end{pmatrix}'$$

Table 4. The mean coordinate vector ($\tilde{\hat{\mathbf{v}}}$) and the mean log transference ($\tilde{\hat{\mathbf{T}}}$) for the cornea of Subject 1.

$$\tilde{\hat{\mathbf{v}}} = \begin{pmatrix} -.0124 & .0001 & -.0000 & -.0000 & -.0005\text{ m} & -.0000\text{ m} & -.0000\text{ m} & -42.0513\text{ D} & 0.0636\text{ D} & 0.0000\text{ D} \end{pmatrix}'$$

$$\tilde{\hat{\mathbf{T}}} = \begin{pmatrix} -0.0123 & -0.0000 & 0.0005\text{ m} & 0\text{ m} \\ -0.0000 & -0.0125 & 0\text{ m} & 0.0005\text{ m} \\ -41.9877\text{ D} & 0.0000\text{ D} & 0.0123 & 0.0000 \\ 0.0000\text{ D} & -42.1148\text{ D} & 0.0000 & 0.0125 \end{pmatrix}$$

(a)



(b)

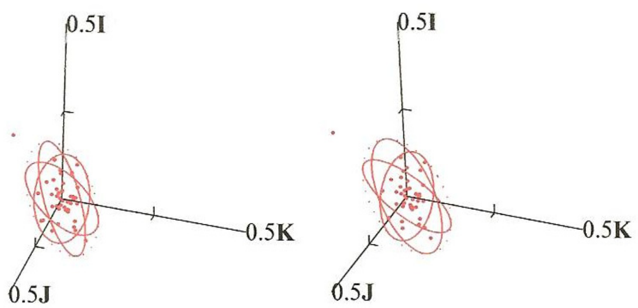


Figure 1. Stereo-pair scatter plots in symmetric power space representing 43 successive Pentacam measurements of the anterior (a) and posterior (b) corneal powers are indicated. The three mutually orthogonal axes are representation of the scalar **I**, ortho-antistigmatic **J**, and oblique antistigmatic **K** components of the powers. Each stereo-pair should be viewed by converging the eyes to a point in front of the figure until a three-dimensional percept is formed. The origin of each stereo-pair represents the sample mean (see Table 1). Each dot represents one measurement and the axes are marked at intervals of 0.25 D. There is more variation in corneal power for the anterior surface (a) than for the posterior surface and 95% surfaces of constant probability density are shown for each sample. One or two points or dots might be regarded as outliers (see the isolated point in plot (b) for example).

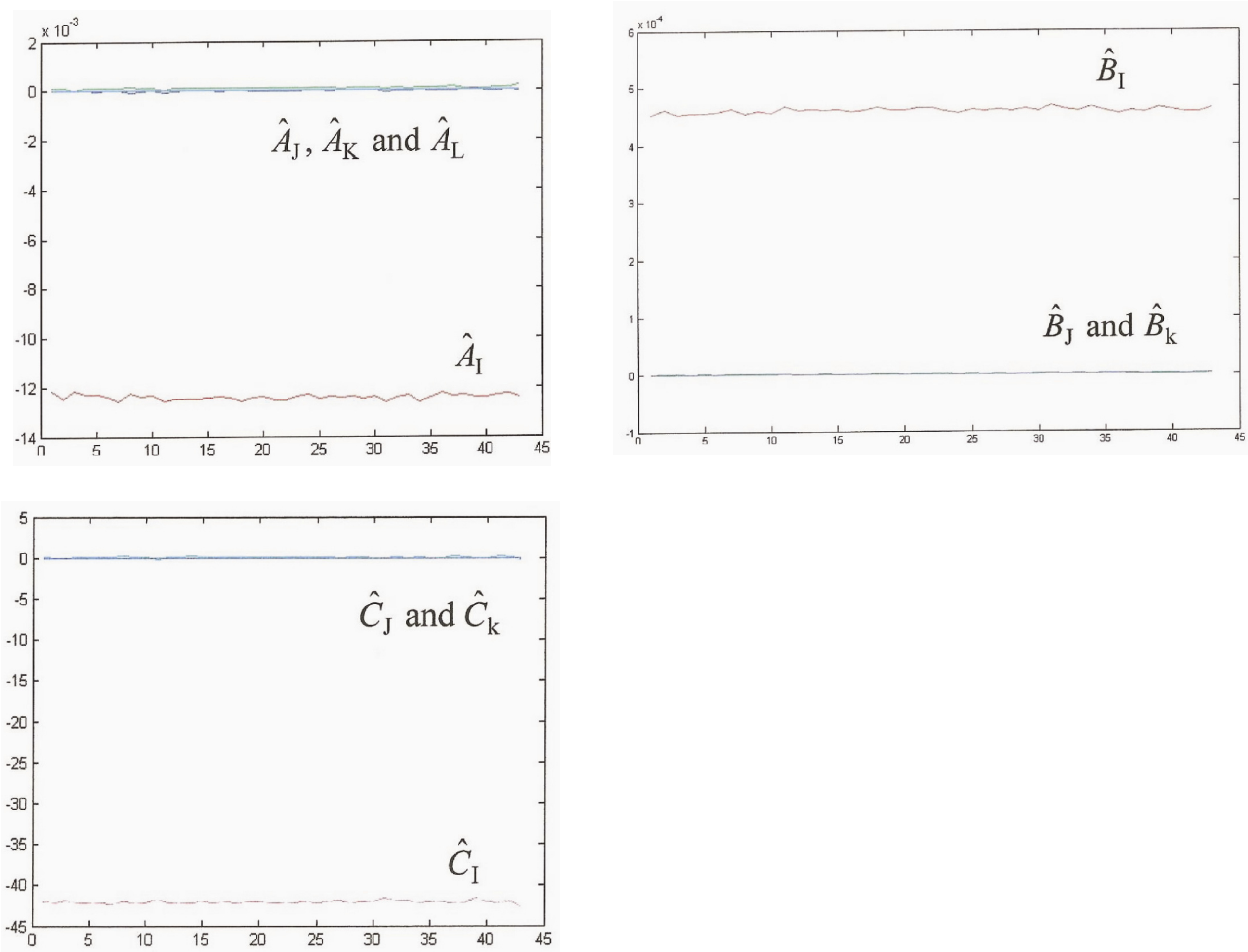


Figure 2. Plots of the 10 components of 43 Hamiltonian coordinate vectors $\hat{\mathbf{v}}$ for the right cornea of Subject 1. Red represents scalar ($\mathbf{I}$) coefficients, green ortho-antistigmatic ($\mathbf{J}$) coefficients, blue oblique antistigmatic ($\mathbf{K}$) coefficients and cyan the antisymmetric ($\mathbf{L}$) coefficients. There is little variation except for the scalar coefficients in red.

Table 5. The sample variance-covariance of the log-transferences of the cornea. Units have been omitted to save space. They are in metres and dioptres where appropriate.

$$\hat{\mathbf{S}} = \begin{pmatrix} 0.0000 & -0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0011 & 0.0003 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0034 & -0.0010 & -0.0000 \\ 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0003 & 0.0004 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0001 & -0.0001 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & -0.0000 & 0.0000 \\ 0.0011 & -0.0034 & -0.0003 & -0.0000 & 0.0001 & 0.0000 & 0.0000 & 29.7165 & -4.2038 & 0.0003 \\ 0.0003 & 0.0010 & 0.0004 & -0.0000 & -0.0001 & -0.0000 & -0.0000 & -4.2038 & 8.0206 & -0.0004 \\ -0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0003 & -0.0004 & 0.0000 \end{pmatrix} \times 10^{-3}$$

Table 6. The mean of transferences ($\tilde{\mathbf{T}}$) of the cornea recovered from the mean of the transformed transference ($\tilde{\tilde{\mathbf{T}}}$).

$$\tilde{\mathbf{T}} = \begin{pmatrix} 0.9782 & -0.0000 & 0.0005\text{ m} & 0.0000\text{ m} \\ -0.0000 & 0.9779 & 0.0000\text{ m} & 0.0005\text{ m} \\ -41.8538\text{ D} & 0.0000\text{ D} & 1.0027 & 0.0000 \\ 0.0000\text{ D} & -41.9801\text{ D} & 0.0000 & 1.0029 \end{pmatrix}$$

Discussion

Basic quantitative analysis of the optical character of thin systems is relatively straight forward. This has been done above for the anterior and posterior surfaces of a cornea. For thick systems (for example, the corneal surfaces plus its central thickness), however, quantitative analysis has been problematic because of symplecticity. Two accompanying papers now develop an approach which performs the quantitative analysis in a 10-dimensional Hamiltonian space. This paper applies the approach to the human cornea, one of the simplest thick systems. In so doing it illustrates application of the approach to systems in general. This paper is in fact the first to do so.

From the set of 43 transferences made on an eye an average transference of the cornea has been determined (Table 6) and variation of the transferences has been indicated via a variance-covariance matrix in log-transference space (Table 5). The fact that most of the entries in the variance-covariance matrix are zero or close to zero is a consequence of the cornea being a system that is what one might call 'almost' thin. Were the cornea a thin system then all entries, with the exception of the lower-right 3×3 block (representing the variation in $\hat{\mathbf{C}}$) would be zero. That all the other entries are not zero reveals the optical character as that of a system that is thick.It is meaningful to bring the mean value back from the log-transference space and report an average transference as we have done here. However, because transferences do not define a vector or linear space the same cannot be done with the variance-covariance; the variance-covariance has to remain in the log-transference space. This is no problem for some purposes (one can, for example, test hypotheses in the log-transference space). But it is inconvenient if one is interested in the variation of the system itself. Ideally one would want to do for transference what plots like those in Figure 1 do for dioptric power. But that is impossible even if it were possible for us to make and visualize stereo-pair pictures representing the 10 dimensions required. One can resort to plots like those in Figure 2 but for components of the transference they cannot give a holistic representation of the system's variation. Preliminary studies suggest that there might be circumstances in which one can define an approximate variance-covariance for the transference which would then give an approximate but holistic measure of variation. These are issues, however, that await further study.

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