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An explicit formula for the matrix logarithm

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Abstract

We present an explicit polynomial formula for evaluating the principal logarithm of all matrices lying on the line segment $\{\mathbf{I}(1-t)+\mathbf{A}t : t \in [0,1]\}$ joining the identity matrix $\mathbf{I}$ (at $t = 0$) to any real matrix $\mathbf{A}$ (at $t = 1$) having no eigenvalues on the closed negative real axis. This extends to the matrix logarithm the well known Putzer's method for evaluating the matrix exponential. A particular application of the matrix logarithm in Optometry is mentioned.

Introduction

Given a nonsingular matrix, $\mathbf{A}\in\mathbb{R}^{n\times n}$, any solution of the matrix equation $e^{\mathbf{X}}=\mathbf{A}$, where $e^{\mathbf{X}}$ denotes the exponential of the matrix $\mathbf{X}$, is called a *logarithm* of $\mathbf{A}$. In general, a non-singular real matrix may have an infinite number of real and complex logarithms. If $\mathbf{A}$ has no eigenvalues on the closed negative real axis then $\mathbf{A}$ has a unique real logarithm with eigenvalues in the open strip $\{z \in\mathbb{C}: -\pi < \mathrm{Im}(z) < \pi\}$ of the complex plane[1]. This unique logarithm may be written as a polynomial in $\mathbf{A}$ and is called the *principal* logarithm of $\mathbf{A}$. It will be denoted by $\log \mathbf{A}$.

The problem of computing the principal matrix logarithm has received some attention in recent years[2-6]. In part, this interest has been motivated by the applications of the matrix logarithm in several scientific areas.

In addition to the applications listed in the above cited papers, we mention the importance that the matrix logarithm has had in Ophthalmic Optics, namely in the recent work of Harris[7, 8] on the study of the average Gaussian eye. Given a set of N eyes with transferences $\mathbf{T}_j$, Harris[7] defines the average eye as an eye with transference

$$\tilde{\mathbf{T}} := \exp\left(\frac{1}{N}\sum_{j=1}^{N}\operatorname{Log}\mathbf{T}_j\right).$$

The method we present in this paper seems to be an important tool for computing and for deriving formulae for the transference $\tilde{\mathbf{T}}$ of the average eye. For 2×2 transferences, Harris[7] uses this method to obtain closed forms for $\tilde{\mathbf{T}}$.

As far as we know, most of the methods proposed for computing the principal logarithm are approximation methods. Unlike the matrix exponential case, for which several closed forms based on polynomial representations have been studied[9-12], little attention has been paid to closed forms for the matrix logarithm.

In this paper, we find for the matrix logarithm the analogue of the well known Putzer's method[12] for evaluating the matrix exponential. Assuming that for $t\in\mathbb{R}$ the spectrum of $\mathbf{I}-\mathbf{A}t$ does not intersect $\mathbb{R}_0^-$, we consider the curve $t\to\log(\mathbf{I}-\mathbf{A}t)$ in $\mathbb{R}^{n\times n}$. Using the coefficients of a polynomial $p(\lambda)$ of degree k such that $p(\mathbf{A})= 0$, every matrix in that curve will be written as a linear combination of the matrices $\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^{k-1}$, in the following way:

$$\log(\mathbf{I}-\mathbf{A}t) = f_1(t)\mathbf{I}+f_2(t)\mathbf{A}+ \cdots +f_k(t)\mathbf{A}^{k-1},$$

where the coefficients $f_1,\ldots, f_k$ are integrals of certain rational functions.

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We find this simple method suitable for teaching purposes because the topics required for understanding it (basically, eigenvalues of matrices and integration of rational functions) are usually taught in the first years of undergraduate courses. We recall that, in contrast, other methods proposed for evaluating the matrix logarithm require advanced theory, such as Schur decompositions, matrix square roots and matrix Padé approximants.

1. A polynomial formula for the matrix logarithm

Given $\mathbf{A} \in \mathbb{R}^{nxn}$, let
$p(\lambda) = \lambda^k + c_1\lambda^{k-1} + \ldots + c_{k-1}\lambda + c_k$
be a polynomial with real coefficients such that $p(\mathbf{A}) = 0$ and let

$$\mathbf{C} = \begin{bmatrix} 0 & \cdots & 0 & -c_k \\ & & & -c_{k-1} \\ & \mathbf{I}_{k-1} & & \vdots \\ & & & -c_1 \end{bmatrix},$$

where $\mathbf{I}_m$ denotes the mxm identity matrix, be the companion matrix of $p(\lambda)$. Examples of polynomials $p(\lambda)$ such that $p(\mathbf{A})=0$ are the characteristic polynomial of $\mathbf{A}$ ($k=n$) and the minimum polynomial of $\mathbf{A}$ ($k \leq n$).

Before stating our main result, let us define the following subset of $\mathbb{R}$:

$$D = \left\{ t \in \mathbb{R} : \sigma(\mathbf{I} - \mathbf{A}t) \cap \mathbb{R}_0^- = \phi \right\},$$

where $\sigma(\mathbf{X})$ stands for the spectrum of $\mathbf{X}$ and $\mathbf{A}$ is a given nxn matrix. For each $t \in \mathbb{R}$, the eigenvalues of $\mathbf{I}-\mathbf{A}t$ are of the form $1-\lambda t$, with $\lambda \in \sigma(\mathrm{A})$. Since non real eigenvalues of $\mathbf{A}$ always give rise to non real eigenvalues of $\mathbf{I} - \mathbf{A}t$, it is enough to consider real eigenvalues of $\mathbf{A}$ to obtain a more clear description of the set D. Thus, we may write ,

$$D = \left\{ t \in \mathbb{R} : 1 - \lambda t > 0, \quad \forall \lambda \in \sigma(\mathbf{A}) \cap \mathbb{R} \right\}.$$

Let $\lambda_M = \max\left(\sigma(\mathbf{A}) \cap \mathbb{R}\right)$ and $\lambda_m = \min\left(\sigma(\mathbf{A}) \cap \mathbb{R}\right)$ Assuming that $\mathbf{A}$ has both positive and negative real eigenvalues, we have $D = \left]1/\lambda_m, \; 1/\lambda_M\right[$. If $\mathbf{A}$ does not have negative eigenvalues then $D = \left]-\infty, \; 1/\lambda_M\right[$ and if $\mathbf{A}$ does not have positive eigenvalues then $D = \left]1/\lambda_m, +\infty\right[$ In any case, D is an open interval.

Theorem 1.1

Suppose that the above notation holds and that the vector function $[f_1(t), \ldots, f_k(t)]^T$ *is the solution in D of the initial value problem*

$$(\mathbf{I}-\mathbf{C}t)\dot{x}(t) = -e_2, \quad x(0) = 0 \tag{1}$$

where $e_2 = [0\ 1\ 0\ \ldots\ 0]^\mathrm{T}$. *Then*

$$\log(\mathbf{I} - \mathbf{A}t) = f_1(t)\,\mathbf{I} + f_2(t)\,\mathbf{A} + \cdots + f_k(t)\mathbf{A}^{k-l}, \tag{2}$$

for all $t \in D$.

Proof. The function $\mathbf{X}(t) = \log(\mathbf{I}-\mathbf{A}t)$ is differentiable for all $t \in D$ and $\dot{\mathbf{X}}(t) = -\mathbf{A}(\mathbf{I}-\mathbf{A}t)^{-1}$ (see (6.6.14) and (6.6.19) in the reference 1). Besides, $\mathbf{X}(t)$ is the unique solution in D of the initial value problem

$$(\mathbf{I}-\mathbf{A}t)\dot{\mathbf{Y}}(t) = -\mathbf{A}, \; \mathbf{Y}(0) = 0 \tag{3}$$

where $\mathbf{Y}(t) \in \mathbb{R}^{nxn}$.

Let $[f_1(t), \ldots, f_k(t)]^T$ be the solution of (1) in D and define

$$p(t) := f_1(t)\mathbf{I} + f_2(t)\mathbf{A} + \ldots + f_k(t)\mathbf{A}^{k-1}.$$

In the following we show that $p(t)$ is also a solution of Equation (3). Clearly $p(0)=0$ because $f_j(0) = 0, \; \forall j = 1, \ldots, k$

Since the vector function $[f_1(t), \ldots, f_k(t)]^T$ satisfies Equation (1), a little calculation lead us to the system

$$\begin{cases} \dot{f}_1 + c_k t \dot{f}_k & = \quad 0 \\ -t\dot{f}_1 + \dot{f}_2 + c_{k-1} t \dot{f}_k & = \quad -1 \\ -t\dot{f}_2 + \dot{f}_3 + c_{k-2} t \dot{f}_k & = \quad 0 \\ \quad \cdots & \\ -t\dot{f}_{k-1} + (1 + c_1 t)\dot{f}_k & = \quad 0 \end{cases} \tag{4}$$

Using the equations of (4) and the identity
$\mathbf{A}^k = -c_1\mathbf{A}^{k-1} - \cdots - c_{k-1}\mathbf{A} - c_k\mathbf{I}$,
which follows from the Cayley-Hamilton theorem, we may write

$$
\begin{aligned}
(\mathbf{I}-\mathbf{A}t)\dot{p}(t) &= (\mathbf{I}-\mathbf{A}t)\,(\dot{f}_1\mathbf{I} + \dot{f}_2\mathbf{A} + \ldots + \dot{f}_k\mathbf{A}^{k-1}) \\
&= \dot{f}_1\mathbf{I} + (\dot{f}_2 - t\dot{f}_1)\,\mathbf{A} + \ldots + \\
&\quad (\dot{f}_k - t\dot{f}_{k-1})\,\mathbf{A}^{k-1} - t\dot{f}_k\mathbf{A}^k \\
&= (\dot{f}_1 + c_k t\dot{f}_k)\mathbf{I} + (-t\dot{f}_1 + \dot{f}_2 + c_{k-1}t\dot{f}_k)\mathbf{A} \\
&\quad + \ldots + (-t\dot{f}_{k-1} + (1-c_1 t)\,\dot{f}_k)\mathbf{A}^{k-1} \\
&= -\mathbf{A}\,.
\end{aligned}
$$

Since (3) has a unique solution, it follows that $p(t) = \log(\mathbf{I}-\mathbf{A}t)$. This concludes the proof .

Since the coefficients functions in (2) are solutions of (4), we can obtain formulae for $\dot{f}_j$, $j = 1,\ldots, k$, by solving the first equation for $\dot{f}_1$ and substituting it into the second equation, solving the second equation for $\dot{f}_2$ and substituting it into the third equation and proceeding similarly until the last equation. The result is

$$\dot{f}_1 = -c_k\dot{f}_k t$$

$$\dot{f}_i = -t^{i-2} - \dot{f}_k\sum_{j=1}^{i} c_{k-i+j}t^j, \quad i = 2,\ldots,k-1,$$

$$\dot{f}_k = \frac{-t^{k-2}}{1+c_1t+\cdots+c_kt^k}$$

or, equivalently,

$$\dot{f}_1 = \frac{c_kt^{k-1}}{1+c_1t+\cdots+c_kt^k}$$

$$\dot{f}_i = \frac{-t^{i-2}-c_1t^{i-1}-\cdots-c_{k-i}t^{k-2}}{1+c_1t+\cdots+c_kt^k}, \quad i = 2,\ldots,k-1$$

$$\dot{f}_k = \frac{-t^{k-2}}{1+c_1t+\cdots+c_kt^k}.$$

We note that the constants arising in the integration process to find f_i, $i = 1,\ldots, k$, can be evaluated according to the identities $f_i(0) = 0$, $i = 1,\ldots, k$.
We now summarize the previous discussion in the next corollary.

Corollary 1.2 *Given* $\mathbf{A} \in \mathbb{R}^{n\times n}$,
let $p(\lambda) = \lambda^k + c_1\lambda^{k-1} + \cdots + c_{k-1}\lambda + c_k$
be a polynomial with real coefficients such that $p(\mathbf{A})=0$. If $D = \{\, t \in \mathbb{R} : \sigma(\mathbf{I}-\mathbf{A}t) \cap \mathbb{R}_0^- = \phi \,\}$ *then for all* $t \in D$

$$\log(\mathbf{I}-\mathbf{A}t) = f_1(t)\,\mathbf{I} + f_2(t)\,\mathbf{A} + \cdots + f_k(t)\,\mathbf{A}^{k-1},$$

where $f_1,\ldots f_k$ *are differentiable functions in D given by*

$$f_1(t) = \int_0^t \frac{c_ks^{k-1}}{1+c_1s+\cdots+c_ks^k}\,ds$$

$$f_i(t) = \int_0^t \frac{-s^{i-2}-c_1s^{i-1}-\cdots-c_{k-i}s^{k-2}}{1+c_1s+\cdots+c_ks^k}\,ds, \quad i = 2,\ldots,k-1$$

$$f_k(t) = \int_0^t \frac{-s^{k-2}}{1+c_1s+\cdots+c_ks^k}\,ds. \tag{5}$$

Remark 1.3 There exists a relationship between the polynomial $p(\lambda)$ and the polynomial

$q(\lambda) = 1 + c_1\lambda + \ldots + c_k\lambda^k$

in the denominator of the functions under integral symbols in (5):

$q(\lambda) = \lambda^k\, p(1/\lambda)$.

Remark 1.4 The indefinite integrals in (5) may be obtained explicitly because we are dealing with rational functions. We note that many calculus textbooks provide methods for evaluating integrals of these kind of functions. Also, symbolic software packages like Mathematica, Maple or Derive are able to compute them.

Remark 1.5 For $\mathbf{A}$ such that $\sigma(\mathbf{A}) \cap \mathbb{R}_0^- = \phi$, equation (2) allows us to find an explicit formula for evaluating the logarithm of all matrices on the line segment joining $\mathbf{I}$ (at t=0) to $\mathbf{A}$ (at t=1): $\{\mathbf{I}(1-t)+\mathbf{A}t : t \in [0,1]\}$.

Indeed,

$$
\begin{aligned}
\log(\mathbf{I}(1-t)+\mathbf{A}t) &= \log(\mathbf{I}-(\mathbf{I}-\mathbf{A})t) \\
&= f_1(t)\,\mathbf{I} + f_2(t)\,(\mathbf{I}-\mathbf{A}) + \cdots \\
&\quad + f_k(t)\,(\mathbf{I}-\mathbf{A})^{k-1}
\end{aligned}
$$

Obviously, this formula, holds not only for all

$t \in [0,1]$, but also for any t such that

$$\sigma\big(\mathbf{I} - (\mathbf{I} - \mathbf{A})t\big) \cap \mathbb{R}_0^- = \phi .$$

In particular, for $t=1$ we may compute directly $\log \mathbf{A}$:

$$\boxed{\log \mathbf{A} = f_1(1)\,\mathbf{I} + f_2(1)\,(\mathbf{I} - \mathbf{A}) + \cdots + f_k(1)\,(\mathbf{I} - \mathbf{A})^{k-1}}.$$

2. Example

To illustrate the method proposed, we consider the matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 4 & -4 \\ 4 & 7 & -4 \\ -1 & -1 & 4 \end{bmatrix}.$$

To compute $\log \mathbf{A}$ we have to work with the matrix $\mathbf{I}-\mathbf{A}$. The spectrum of $\mathbf{I}-\mathbf{A}$ is $\{-11,-2,-2\}$ and its minimum polynomial is $p(\lambda)=\lambda^2+13\lambda+22$. Applying directly (5), we have

$$f_1(t) = \int_0^t \frac{22s}{1+13s+22s^2}\, ds$$

$$f_2(t) = \int_0^t \frac{-1}{1+13s+22s^2}\, ds.$$

Evaluating the integrals, we may write

$$\log\big(\mathbf{I} - (\mathbf{I} - \mathbf{A})t\big) = f_1(t)\,\mathbf{I} + f_2(t)\,(\mathbf{I} - \mathbf{A}),$$

where $t \in [0,1]$ and

$$f_1(t) = \frac{11}{9}\ln(1+2t) - \frac{2}{9}\ln(1+11t)$$

$$f_2(t) = \frac{1}{9}\ln\left(\frac{1+2t}{1+11t}\right).$$

Therefore

$$\begin{aligned}\log(\mathbf{A}) &= f_1(t)\,\mathbf{I} + f_2(t)\,(\mathbf{I} - \mathbf{A}) \\ &= \left(\ln 3 + \frac{2}{9}\ln\left(\frac{1}{4}\right)\right)\mathbf{I} + \frac{1}{9}\ln\left(\frac{1}{4}\right)(\mathbf{I} - \mathbf{A}).\end{aligned}$$

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