The Jackson Cross-Cylinder. Part 1: Properties

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Abstract

The Jackson cross-cylinder is a lens of fundamental importance in optometry with a key role in the refraction routine. And yet it appears not to be as well understood as perhaps it should be. The purpose of this paper is to examine the linear optical character of the Jackson cross-cylinder and, in particular, those properties associated with the operations performed on the lens in the refraction routine, namely flipping and turning. Corresponding to these operations in physical space are steps in an abstract space, symmetric dioptric power space. The powers of all Jackson cross-cylinders lie in the plane of antistigmatic powers in the abstract space. In particular the powers of an F Jackson cross-cylinder (for example, a 0.5-D Jackson cross-cylinder has $F = 0.5$ D) lie on a circle of radius F centred on null power. Flipping the lens takes one diametrically across the circle; turning the lens takes one around the circle at twice the rate. A subsequent paper shows how these operations work in defining the cylinder in the refraction routine.

Key words: Jackson cross-cylinder, crossed cylinder, antistigmatic power, symmetric dioptric power space, dioptric power matrix

Introduction

In the refraction routine the refractionist follows a standard procedure making use of the Jackson cross-cylinder to define the cylindrical component of the refractive error. Despite this everyday use of the lens there does not seem to be much understanding of how the lens works. Accordingly the purpose of this paper is to take a close look at the linear optics of the Jackson cross-cylinder and so to provide a basis for a subsequent paper[1] in which the clinical use of the lens is examined.

Despite its common use as a clinical tool, and the fact that it gets to the very nitty-gritty of the nature of astigmatism, the Jackson cross-cylinder is undervalued in optometry. That optometry traditionally thinks of astigmatism in terms of cylinder, instead of the Jackson cross-cylinder, is a mistake that costs the profession dear. The arrival of new technologies (refractive surgery and wavefront analysis, for example), though, is beginning to force reassessment of the nature of astigmatism and is likely to bring the Jackson cross-cylinder into greater prominence. In the meantime modern developments, including the dioptric power matrix and symmetric dioptric power space, in particular, provide a good model for understanding the Jackson cross-cylinder, as we shall see below. We begin at the beginning, with Stokes and Jackson; we look at some of the terminological difficulties; we then examine the Jackson cross-cylinder in detail and the physical operations that are performed on it in the clinical routine; and we see the corresponding steps that occur in the abstract space of symmetric dioptric powers. The accompanying paper[1] then shows how the operations on the Jackson cross-cylinder combine to work together in the refraction routine.

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A Little History

Perhaps the earliest description of the lens commonly known as the Jackson cross-cylinder is that[2, 3] by Stokes in 1849: "Conceive a lens ground with two cylindrical surfaces of equal radius, one concave and the other convex, with their axes crossed at right angles; call such a lens an *astigmatic lens*; … and a line parallel to the axis of the convex surface the *astigmatic axis*." (The reader may wish to look ahead at Figure 3.) Stokes goes on to represent his astigmatic lens as a vector and to show that the combination of two astigmatic lenses is equivalent to an astigmatic lens whose character can be determined as the sum of the vectors. He points out that "a sphero-cylindrical lens is equivalent to a common lens, the power of which is equal to the semi-sum of the reciprocals of the focal lengths in the two principal planes, combined with an astigmatic lens, the power of which is equal to their semi-difference."

Notice that, in Stokes's terminology, an astigmatic lens necessarily has principal powers F_1 and F_2 such that $F_2 = -F_1$. Thus a spherocylindrical lens is *not* an astigmatic lens, unless it happens to have $F_2 = -F_1$. A spherocylindrical lens, however, is equivalent to the combination of a common (usually called spherical) lens and an astigmatic lens. Also notice that Stokes defines the astigmatic axis of the astigmatic lens; the astigmatic axis is *parallel* to the axis of the positive cylinder and is the same as the principal meridian along which the power is negative. Stokes's vectorial sums are equivalent to the sums of dioptric power matrices in the antistigmatic plane that we shall refer to below. Stokes thinks of astigmatism as the *semi*-difference of principal powers not the difference, that is, half the cylinder as it were and not the cylinder itself.

In 1886 Jackson enthuses about Stokes's astigmatic lens and tells how it has become an essential element in his trial case[4]: "The two used in this case are:

-0.25 sph. $\bigcirc$ $+0.50$ cyl.,

-0.50 sph. $\bigcirc$ $+1$ cyl.,

of which the former is the most generally useful. For two years I have employed such a lens to hold in front of the approximate correction, to determine if a cylindrical lens, or a modification of the cylindrical lens[,] already chosen will improve it; and it is far more useful, and far more used, than any other lens in my trial set."

By 1907 Jackson[5] is as enthusiastic as ever about Stokes's lens. Oddly, though, he is now calling it an *astigmic lens*; he uses the astigmic lens "to determine the amount and principal meridians of astigmia". He also calls it a *crossed cylinder*. Interestingly Jackson makes a passing comment that both Stokes and Dennett[6] had "proposed to use it alone, letting it take the place of the cylindrical lenses usually found in the trial case." Then in 1930 we find that Jackson[7] has dropped the terms *astigmia* and *astigmic* for *astigmatism* and *astigmatic*. And the lens he is calling a *cross cylinder*.

While Stokes[2, 3] saw clearly that it was his astigmatic lens that represented the real essence of astigmatism it seems that ophthalmology persisted in regarding astigmatism as cylinder and the crossed cylinder as merely a tool for finding cylinder. Optometry has followed suit. For example, according to Bennett and Rabbetts[8] "The astigmatism … may be expressed in dioptres as the difference [as opposed to *semi*-difference] between the two principal powers."

Dennett seems to have been an exception among ophthalmologists. He comments[6] that ophthalmology regarded Stokes's lens "as little better than an interesting toy". However I say 'seems' because of the statement he goes on to make about "one difficulty" associated with the lens. The difficulty concerns the writing of the prescription from the readings on the lens in front of the eye. He gives an example in which the lens indicates a cylinder of power 2 D and "+ axis at 15° … and a spherical +3 behind the Stokes' lens". He explains that, "Owing to the spherical element which always enters into the Stokes' lens combination, the spherical glass which was used behind the [Stokes' lens] must not be used in the prescription but it must be increased or decreased … by a glass just half as strong as the cylinder". Hence, in this case, 'the prescription may be written:

+2. D. cyl. 15° $\bigcirc$ (–1. sph. $\bigcirc$ +3. sph.)

which, of course, is equal to

+2. D. cyl. 15° $\bigcirc$ +2. sph.'

Dennett's statement seems to come across as a negative comment on Stokes's lens rather than, as it should be, a negative comment on ophthalmology's concept of astigmatism.

Terminology

In addition to conceptual differences there are terminological difficulties. Jackson's contribution was not in inventing the Jackson cross-cylinder; Stokes, apparently, should get the credit rather than Jackson. Jackson, however, does deserve the credit for making Stokes's lens part of the refraction routine. So we should really talk of Jackson's routine and refer to the fact that it makes use of Stokes's lens.

Bennett and Rabbetts[8] comment that "in manufacture, an equivalent sphero-cylindrical form is preferred for practical convenience. The recommended term cross cylinder is thus more appropriate than crossed cylinder." The logic here seems not entirely clear; for a start there are two cylinders and not just one. But one does not want to get hung up on this point. In order to get on, and despite the terminological inappropriateness, we shall use the term *Jackson cross-cylinder*. The hyphen seems preferable (Tunnacliffe[9] uses it) because, in combinations (for example, *cross-bar*, *cross-bow*, *cross-road*, *crossword*), *cross* is not usually disconnected from what follows.

I have considerable difficulty with Stokes's use of the word *astigmatic*. It seems to me that all sphero-cylinders that are not stigmatic are necessarily astigmatic. So his use is in a very restricted sense of the term. Because of the importance of the lens and the concept I believe we do need a good word that means astigmatic in Stokes's narrower sense. The best I have been able to come up with is *antistigmatic*[10]. An *a*stigmatic lens is merely a lens that is not stigmatic. An *anti*stigmatic lens is a special lens that is equal-but-opposite as it were in two orthogonal meridians; it is as different from a stigmatic lens as a lens can be. *Anti*- is suggested partly by the requirement that $F_2 = -F_1$. It is also suggested by parallel terminology in mathematics: symmetric, *a*symmetric and *anti*symmetric matrices.

In our terminology, then, the Jackson cross-cylinder is an antistigmatic lens.

We turn now to the Jackson cross-cylinder itself.

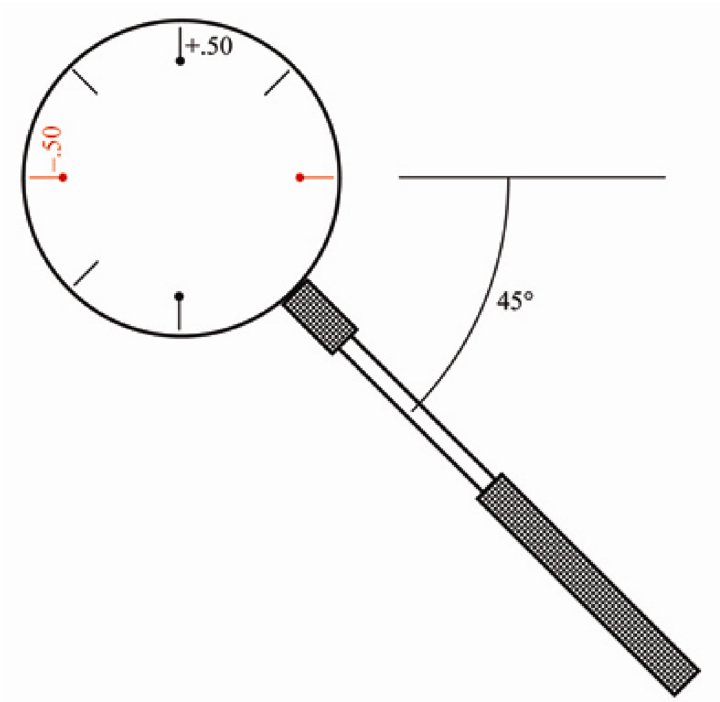

Figure 1 A 0.5-D Jackson cross-cylinder in primary orientation. The handle is down to the right and at 45º to the horizontal. The label +.50 is in the usual orientation for reading; –.50 reads upward. The markings are in red (as here) and white (shown black here). The lens has a front or first surface, the surface facing the reader; the markings are on this surface. The second or back surface of the lens is the surface facing away from the reader. The lens has power given in various representations in Table 1. The two red pins represent the *axis* of the component cylinder of power –0.5 D; the so-called power meridian of that cylinder is vertical (see Figure 4(a)). The two white pins represent the axis of the component cylinder of power 0.5 D. When the lens is flipped the back surface comes to the front and the markings are reversed; in particular the numbers (–.50 and +.50) appear as their mirror images.

The Jackson Cross-Cylinder in Primary Orientation

Figure 1 shows a Jackson cross-cylinder. It happens to be one in the writer's possession. The markings on the lens are in red and white. The white markings are represented here in black. (The reader is cautioned that markings are not consistent across all lenses; the description here might need modification in the case of lenses with different markings.) This particular lens is a 0.5-D Jackson cross-cylinder.

The lens can be held in various orientations. The particular orientation shown in Figure 1 is what we shall call the *primary* orientation of the lens. In the primary orientation the label +.50 is in the usual orientation for reading; that is, +.50 is horizontal, right way up and reads from left to right. The red label –.50 reads upward. The markings are on the front surface of the lens (the surface facing the reader).

Inspection shows that the lens in Figure 1 is plano-toric; that is, the front surface is flat and the back surface mixed toric or saddle-shaped as illustrated in Figure 2. In Figure 2 the curvature of the back surface has been greatly exaggerated for clarity; and the lens is drawn cut square. We note that, despite its name, neither surface of the lens is cylindrical.

A lens both of whose surfaces are cylindrical is illustrated in Figure 3; it is a bicylindrical lens. It is in fact Stokes's lens[2, 3]. One cylinder has a vertical axis and defines the convex front surface (a). ('Axis' here means the axis of the cylinder in the ordinary geometrical sense.) The other cylinder has a horizontal axis (red) and defines the back surface (b). In the figure the lens is represented as a fusion (c) of a front portion (a) and a back portion (b). (d) shows the lens from the front. We note that the axes of the cylinders lie behind the lens, some 20 mm behind if we take the drawings literally. As in Figure 2 the curvatures are greatly exaggerated; calculation shows that the true distance is actually 1.046 m (for an assumed index of 1.523). Stokes's astigmatic axis is the vertical axis in Figure 3(d) but regarded as lying in the plane of the lens.

If thickness can be neglected then in linear optics the bicylinder of Figure 3 is optically equivalent to the plano-toric of Figures 1 and 2. Being optically equivalent under the stated conditions the two lenses have the same dioptric power.

Table 1 Some representations of the powers of Jackson cross-cylinders.

		Power of Lens in		
		Figure 1	**Figure 6(b)**	**Figure 6(c)**
crossed cylinder		$-0.50\times180\,/+0.50\times90$	$-0.50\times90\,/+0.50\times180$	$-0.50\times30\,/+0.50\times120$
principal meridional	a	$-0.50\text{ al }90\,/+0.50\text{ al }180$	$-0.50\text{ al }180\,/+0.50\text{ al }90$	$-0.50\text{ al }30\,/+0.50\text{ al }120$
	b	$-0.5\{90\}\ 0.5\{180\}$	$-0.5\{180\}\ 0.5\{90\}$	$-0.5\{30\}\ 0.5\{120\}$
	c	$-0.5\{90\}0.5$	$-0.5\{180\}0.5$	$-0.5\{30\}0.5$
	d	$0.5\{180\}-0.5$	$0.5\{90\}-0.5$	$0.5\{120\}-0.5$
spherocylindrical (negative cylinder)		$+0.50/-1.00\times180$	$+0.50/-1.00\times90$	$+0.50/-1.00\times120$
spherocylindrical (positive cylinder)		$-0.50/+1.00\times90$	$-0.50/+1.00\times180$	$-0.50/+1.00\times30$
dioptric power matrix **F**		$0.5\mathbf{J}$ D	$-0.5\mathbf{J}$ D	$\begin{pmatrix} 0.250 & 0.433 \\ 0.433 & -0.250 \end{pmatrix}$ D
coordinate vector $\begin{pmatrix} F_\mathrm{I} & F_\mathrm{J} & F_\mathrm{K} \end{pmatrix}'$		$\begin{pmatrix} 0 & 0.5 & 0 \end{pmatrix}'$ D	$\begin{pmatrix} 0 & -0.5 & 0 \end{pmatrix}'$ D	$\begin{pmatrix} 0 & 0.250 & 0.433 \end{pmatrix}'$ D
power vector[23] (M, J_0, J_{45})		$(0, 0.5, 0)$ D	$(0, -0.5, 0)$ D	$(0, 0.250, 0.433)$ D

The power of the lens in Figure 1 is listed in Table 1 in several ways. The way most closely related to the markings on the lens is the crossed-cylinder representation of power, namely $-0.50\times180\,/+0.50\times90$. This notation is most closely related, not to the actual Jackson cross-cylinder itself, the plano-toric lens in Figure 1, but to the optically-equivalent bicylindrical lens in Figure 3.

The first surface of the lens in Figure 3 has power 0.5 D along the principal meridian at $180°$ and 0 D along $90°$. We write this as the principal meridional power $0.5\{180\}0$. We represent a refracting surface by a vertical line $|$. In particular we represent a surface of power $0.5\{180\}0$ by $|0.5\{180\}0$. The lens in Figure 3 has form $|0.5\{180\}0\,|-0.5\{90\}0$. By contrast the Jackson cross-cylinder of Figure 1 has form $|0\,|0.5\{180\}-0.5$. (This notation is defined elsewhere.[11])

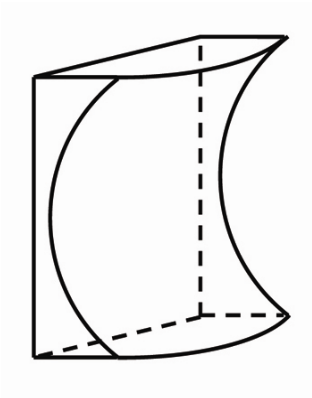

Figure 2 Representation of the lens in Figure 1 turned so that the front surface is facing to the left and away from the reader. The lens is now shown as cut square. The surface curvatures are greatly exaggerated. The reader is looking directly at the back surface and can see the front surface through the lens. The front surface is flat; the back surface is saddle-shaped. The lens has form $|0\,|0.5\{180\}-0.5$.

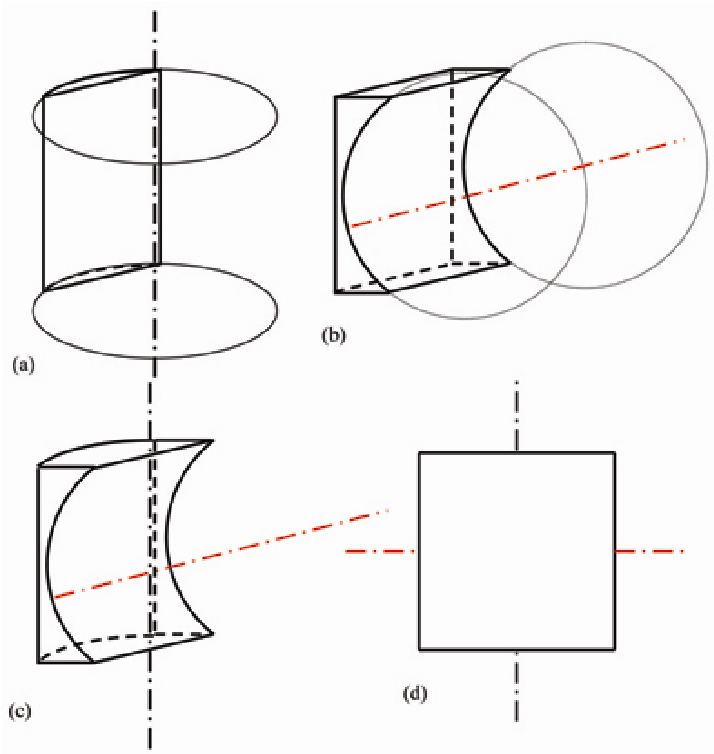

Figure 3 Schematic representation of the bicylindrical lens optically equivalent to the lens in Figures 1 and 2. The lens is imagined made up of a front portion (a) and a back portion (b) fused to make the lens in (c) and shown in front view in (d). Its form is $|0.5\{180\}0\,|-0.5\{90\}0$. The axis of the back cylinder is shown in red; the axis of the front cylinder is shown in black. Notice that both axes lie behind the lens as shown in (d). The curvatures are greatly exaggerated; as a result the axes are shown much closer to the lens in (c) than they actually are, their true positions being just over 1 m behind the lens.

Properties of the Jackson Cross-Cylinder

A number of properties of the Jackson cross-cylinder are illustrated in Figure 4. The principal meridians of its power are shown in (a); the principal meridian of power 0.5 D is horizontal and the power along the vertical meridian is –0.5 D. The vertical meridian is Stokes's astigmatic axis[2, 3]. Distant objects viewed through the lens appear magnified horizontally and minified vertically (Figure 4(b)). Objects exhibit apparent contra-motion when the lens is moved sideways and co-motion when the lens is moved vertically. (The lens also exhibits a scissors movement as described briefly below.) Lines of constant thickness are shown in (c); the thinnest points on the lens are the points labelled A and the thickest points are labelled B. The deflectance vector field (the prismatic effect vector field) is represented in (d). Locally each vector is orthogonal to the thickness contour[12] and points in the direction in which the thickness increases fastest. Equations for thickness, thickness contours and deflectance are given in the Appendix.

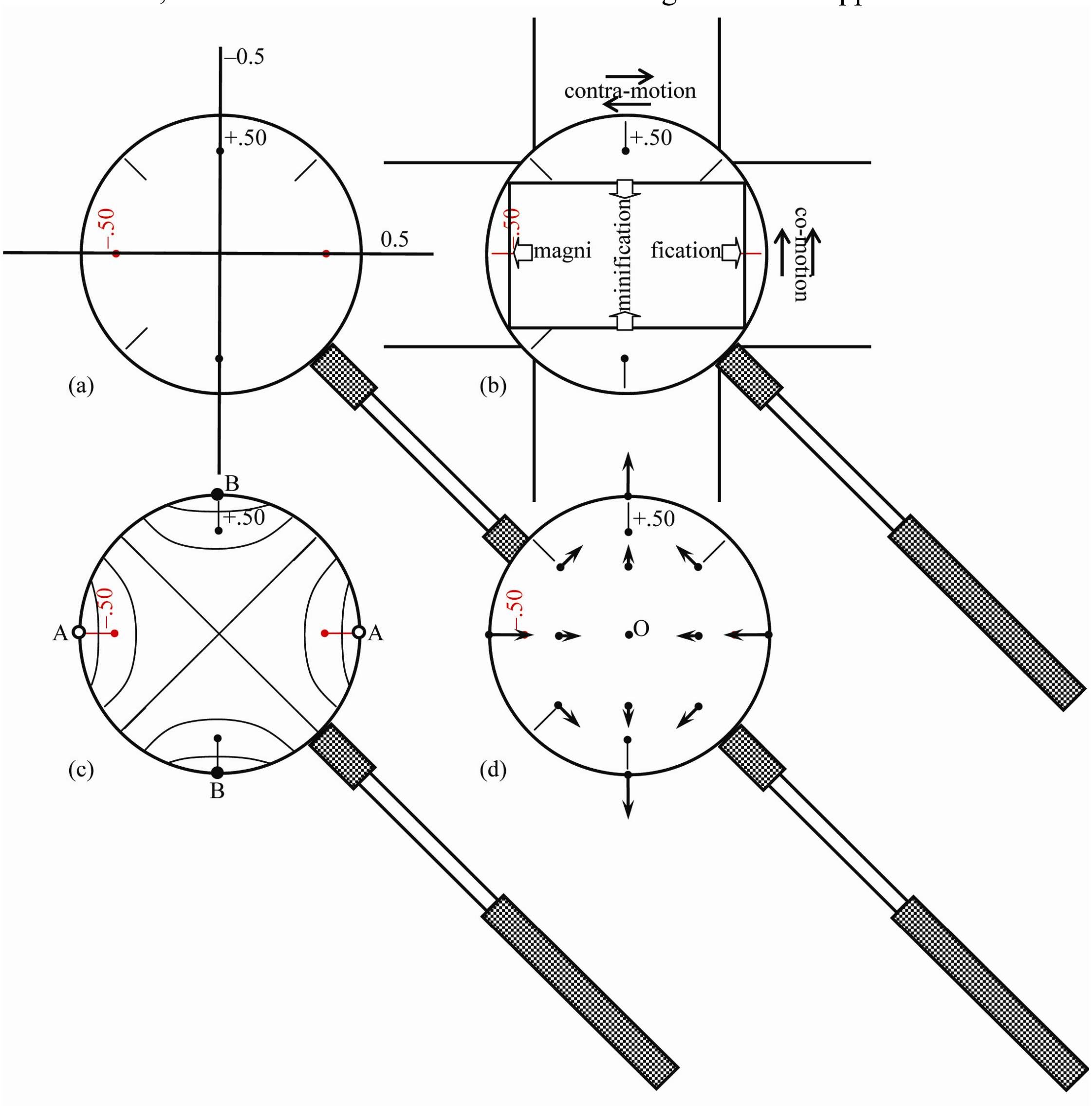

Figure 4 Some properties of the Jackson cross-cylinder in Figure 1. (a) The principal meridians (horizontal and vertical) of the lens and the principal powers along them. Notice that the principal power along the horizontal meridian is 0.5 D. That is in contrast to the fact that the horizontal is labelled –0.5 on the lens. (b) Objects viewed through the lens appear magnified horizontally and minified vertically. Objects exhibit apparent contra-motion when the lens is displaced horizontally and co-motion when the lens is displaced vertically. (c) Approximate lines of constant thickness of the lens. The lens is thinnest at points A and thickest at points B (compare Figure 3). Equations are presented in the Appendix. (d) The approximate deflectance (prismatic effect) vector field for the lens. O is the optical centre; it coincides with the geometrical centre. On the axis of the handle the deflectance is orthogonal to the handle. Equations are presented in the Appendix.

Operations Performed on the Jackson Cross-Cylinder in the Clinical Routine

In addition to the two cylinder axes there are two other axes of importance associated with a Jackson cross-cylinder. Figure 5 shows all four axes. C^+ and C^- are the axes of the two cylinders; they lie behind the lens parallel to the plane of the figure. (Stokes's astigmatic axis is parallel to C^+ and in the plane of the figure.) The optical axis O is orthogonal to the plane of the figure and intersects the lens at its geometric centre. The optical and geometric centres coincide. The handle of the lens defines the flip axis F.

Two operations are performed clinically on the lens: flip and turn. They are illustrated in Figure 6. The lens is flipped about F from (a) to (b) and turned about O from (a) to (c). The rotation is through 180° in the case of flip and through any angle ω (represented here as 30°) in the case of turn. ω is positive for anticlockwise turn and negative for clockwise turn. Flipping the lens takes the front surface to the back. More importantly the operation makes the two principal meridians change places; the meridian of positive power, horizontal in (a), becomes vertical in (b), and the meridian of negative power becomes horizontal. The powers of the lens after the two operations are given in Table 1. The scissors effect observed through the lens when the lens is turned is illustrated in Figure 7.

The handle is horizontal in Figure 7(b). When displaced horizontally and vertically in this orientation objects viewed through the lens appear to exhibit the cross-motion represented in Figure 8.

We turn now to the dioptric power matrix in terms of which the Jackson cross-cylinder and the operations on it are conveniently described.

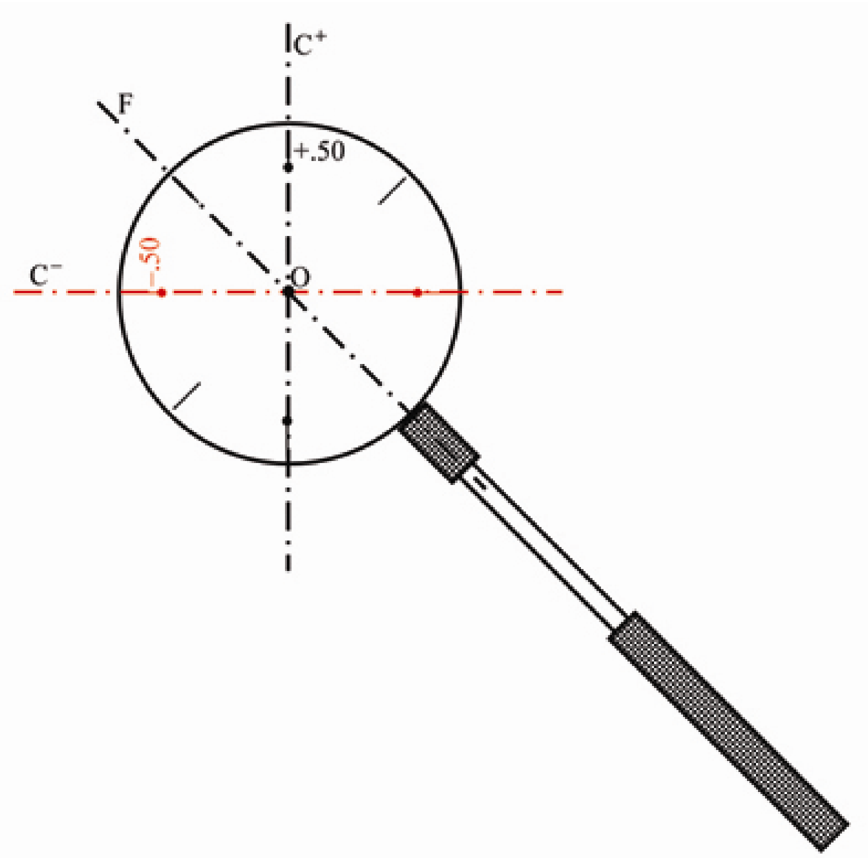

Figure 5 Four axes of the Jackson cross-cylinder. F is the flip axis; it is defined by the handle and lies in the plane of the figure. O is the optical axis; it is orthogonal to the plane of the figure and through the geometrical centre of the lens. C^+ is the axis of the positive cylinder (vertical here) and C^- the axis of the negative cylinder of the equivalent bicylindrical form; the two axes are just over 1 m behind and parallel to the plane of the figure. Stokes's astigmatic axis is C^+ except that it lies in the plane of the lens.

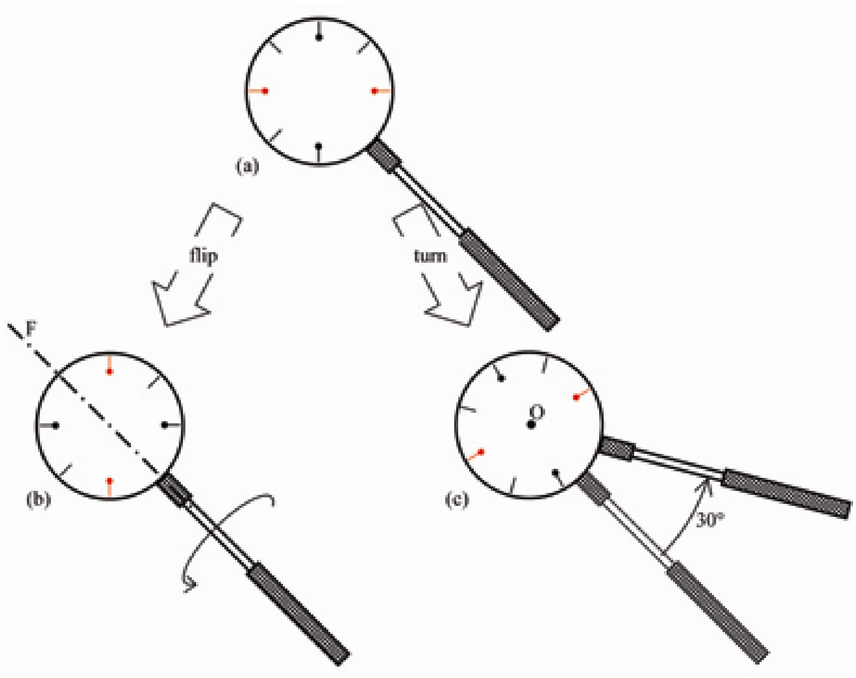

Figure 6 The two operations performed clinically on a Jackson cross-cylinder. (a) A Jackson cross-cylinder oriented as in Figure 1. (b) The Jackson cross-cylinder of (a) has been flipped about the flip axis F; the lens undergoes a rotation of 180 °. The back surface is brought to the front and vice versa. The red and white pins change places. The projections of the two cylinder axes C^+ and C^- change places. In space the axes are now a little more than 1 m above the plane of the figure. The principal meridians change places. (c) The lens in (a) has been turned through 30° about the optical axis O. The powers of the lenses are listed in Table 1.

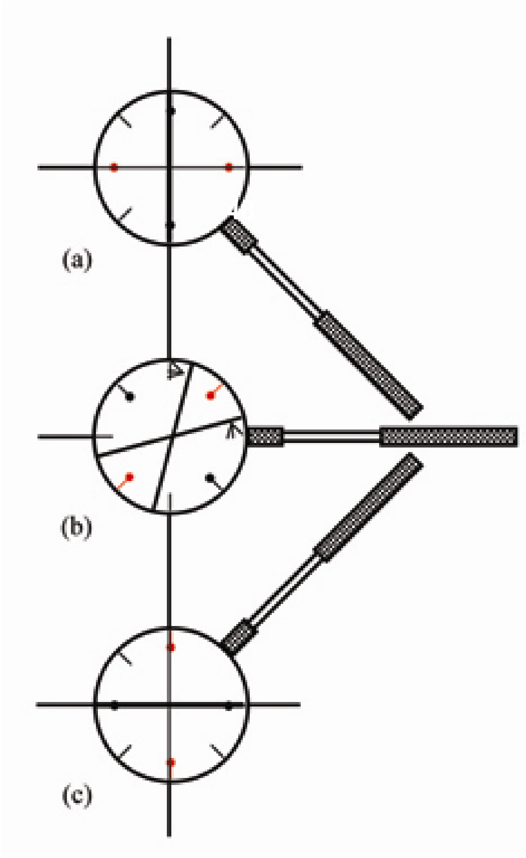

Figure 7 The scissors effect observed when a distant cross is viewed through the Jackson cross-cylinder. The lens is rotated anticlockwise from the primary position in (a) to 45° in (b) and to 90° in (c). The upright of the cross appears initially to rotate in the opposite sense (clockwise) until it reaches an extreme orientation in (b) after which it rotates in the same sense (anticlockwise) back to the vertical in (c). The cross-piece exhibits the opposite behaviour. The thickness of the upright is magnified in (a) and minified in (c), the opposite being the case for the cross-piece.

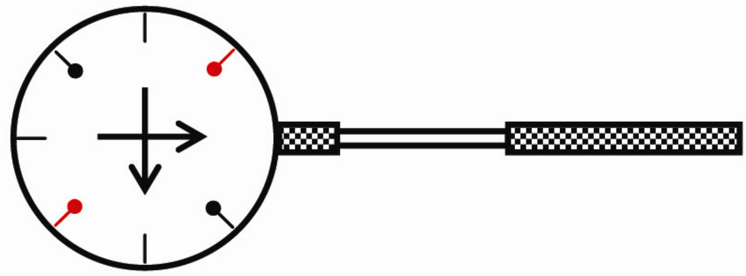

Figure 8 Apparent cross-motion of objects viewed through the lens. Moving the lens in the direction that the handle points (to the right here) makes objects viewed through the lens appear to move vertically downward. Moving the lens downward (in a direction perpendicular to the handle) causes objects to appear to move to the right.

The Dioptric Power Matrix and the Jackson Cross-Cylinder

In its most general form the dioptric power of an optical system is defined in terms of the system's ray transference[13]. It is represented by the 2×2 matrix

$$\mathbf{F} = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}. \qquad (1)$$

For thin systems $\mathbf{F}$ is symmetric $(f_{12} = f_{21})$ and the entries in the matrix can be calculated from equations published by Fick[14-17] and, later, independently by Long[18]: if the spherocylindrical power is $F_\mathrm{S}\ F_\mathrm{C} \times A$ then

$$f_{11} = F_\mathrm{S} + F_\mathrm{C} \sin^2 A \qquad (2)$$

$$f_{12} = f_{21} = -F_\mathrm{C} \sin A \cos A \qquad (3)$$

$$f_{22} = F_\mathrm{S} + F_\mathrm{C} \cos^2 A \quad . \qquad (4)$$

The trace of $\mathbf{F}$ is

$$\mathrm{tr}\mathbf{F} = f_{11} + f_{22}. \qquad (5)$$

Substituting from equations 2 and 4 we obtain

$$\mathrm{tr}\mathbf{F} = 2F_\mathrm{S} + F_\mathrm{C}. \qquad (6)$$

For a Jackson cross-cylinder the principal meridional power is $F\{A\} -F$ and the spherocylindrical power is $F\ -2F \times A$. It follows from equation 6 that a Jackson cross-cylinder has

$$\mathrm{tr}\mathbf{F} = 0. \qquad (7)$$

It is characteristic, then, of the dioptric power matrix $\mathbf{F}$ of a Jackson cross-cylinder that $\mathbf{F}$ is symmetric and has zero trace. (The assumption is made throughout this paper that the Jackson cross-cylinder is thin.)

It is convenient to expand a dioptric power matrix as

$$\mathbf{F} = F_\mathrm{I}\mathbf{I} + F_\mathrm{J}\mathbf{J} + F_\mathrm{K}\mathbf{K} + F_\mathrm{L}\mathbf{L} \qquad (8)$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{L} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \qquad (9)$$

(For more mathematical detail the reader is referred elsewhere[19-21] and to papers cited there.) Substituting equations 9 into equation 8 and solving we obtain the coefficients in equation 8 as semi-sums and semi-differences:

$$F_{\mathrm{I}} = \tfrac{1}{2}\left(f_{11} + f_{22}\right) \tag{10}$$

$$F_{\mathrm{J}} = \tfrac{1}{2}\left(f_{11} - f_{22}\right) \tag{11}$$

$$F_{\mathrm{K}} = \tfrac{1}{2}\left(f_{21} + f_{12}\right) \tag{12}$$

$$F_{\mathrm{L}} = \tfrac{1}{2}\left(f_{21} - f_{12}\right). \tag{13}$$

This echoes Stokes's reference to semi-sums and semi-differences mentioned above.

For thin systems $F_{\mathrm{L}} = 0$ (because of symmetry) and equation 8 reduces to

$$\mathbf{F} = F_{\mathrm{I}}\mathbf{I} + F_{\mathrm{J}}\mathbf{J} + F_{\mathrm{K}}\mathbf{K}. \tag{14}$$

Because of equation 7 this reduces further to

$$\mathbf{F} = F_{\mathrm{J}}\mathbf{J} + F_{\mathrm{K}}\mathbf{K} \tag{15}$$

for Jackson cross-cylinders.

The set of all dioptric powers constitutes what is called *dioptric power space*[13]. Consider the special power with $F_{\mathrm{I}} = 1$ D and all the other coefficients equal to 0 D. The dioptric power matrix is **I** D. It is the power of the thin lens commonly called a 1-D sphere. The four special powers **I** D, **J** D, **K** D and **L** D are called the *basic* powers of dioptric power space. They are said to *span* the space. Because there are four of them the space is called *four-dimensional*.

Thin systems have powers that define a sub-space of dioptric power space called *symmetric dioptric power space*. Three basic powers, **I** D, **J** D and **K** D span symmetric dioptric power space which, then, is three dimensional (equation 14). Two basic powers, **J** D and **K** D, span the space (two-dimensional) of the powers of Jackson cross-cylinders (equation 15). A single basic power, **I** D, spans the space (one-dimensional) of the powers of thin stigmatic lenses, the thin spherical lenses.

Talk here of *space* is a pure mathematical abstraction. There is no connection with space in the familiar sense of the term. However because symmetric dioptric power space is three-dimensional (it is also Euclidean) it is very useful to be able to map it onto physical space and so construct drawings that represent thin lenses and Jackson cross-cylinders in particular. This is what we now do.

Graphical Representation of Symmetric Dioptric Power Space and the Jackson Cross-Cylinder

Because an inner-product is defined on dioptric power space distances and angles are defined in the spaces.[20] This allows one to construct drawings of the spaces, Figure 9 for example. All powers of the form $F_{\mathrm{I}}\mathbf{I}$ are marked off along an axis which we can refer to as axis **I**. Points representing the powers $-2\mathbf{I}$ D, $-\mathbf{I}$ D, **O** D, **I** D and $2\mathbf{I}$ D in particular are labelled in Figure 9(a). The same can be done with axes **J** and **K**. Calculation shows that the angle between any two of these three axes is $90°$ and that **I** D, **J** D and **K** D are all at the same distance, namely 1 D, from **O** D. Because of the orthogonality and the unit length of the basic powers the coordinate system is said to be *orthonormal*.

Figure 9(a) is a graphical representation of symmetric dioptric power space. The powers of all thin systems (symmetric) can be plotted in the space according to equation 14. In particular because they all have $F_{\mathrm{I}} = 0$ Jackson cross-cylinders all lie in the plane (equation 15) perpendicular to axis **I**. This is the plane of antistigmatic powers. The representation of the space is tilted in Figure 9(b) so that we are looking directly down axis **I** at the plane of antistigmatic powers. For the power **J** D the principal meridional power is $1\{180\}-1$ and the spherocylindrical power is $1\ -2\times 180$. For the power **K** D the principal meridional power is $1\{45\}-1$ and the spherocylindrical power is $1\ -2\times 45$.

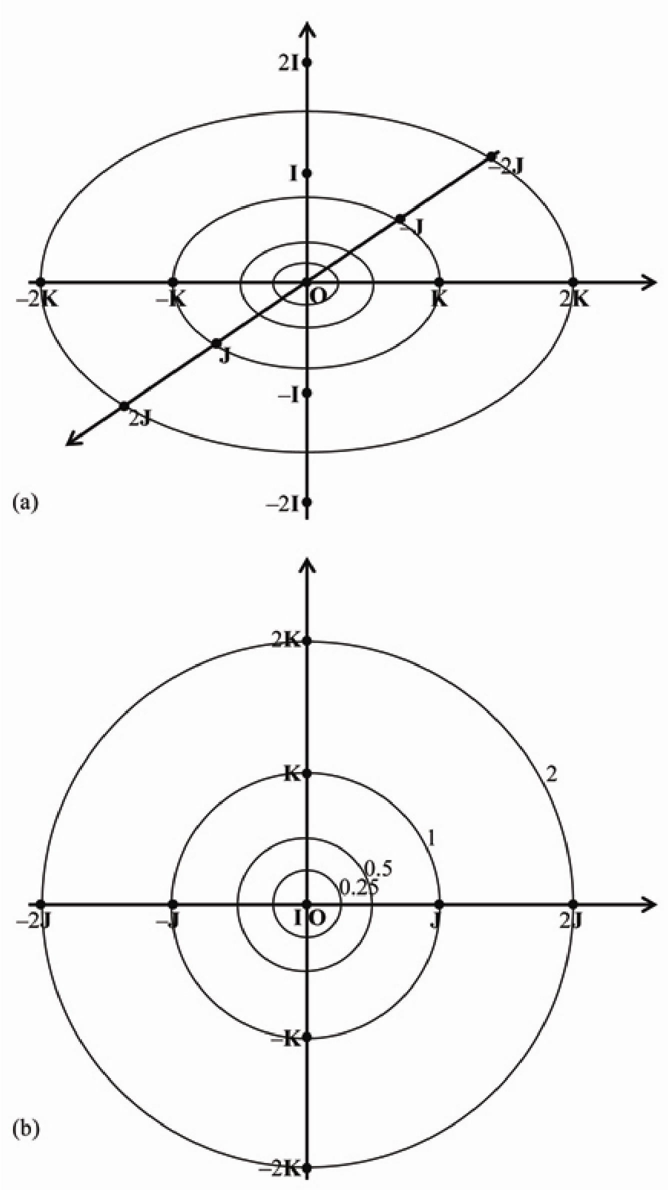

Figure 9 Graphical representation of the abstract space known as symmetric dioptric power space with circles representing the powers of 0.25-D, 0.5-D, 1-D and 2-D Jackson cross-cylinders. The space is tilted in (a) to show the three axes **I**, **J** and **K**. In (b) axis **I** is orthogonal to the plane of the figure and points towards the reader; **J** and **K** span the antistigmatic plane and are in the plane of the figure. The antistigmatic plane represents the powers of all possible Jackson cross-cylinders.

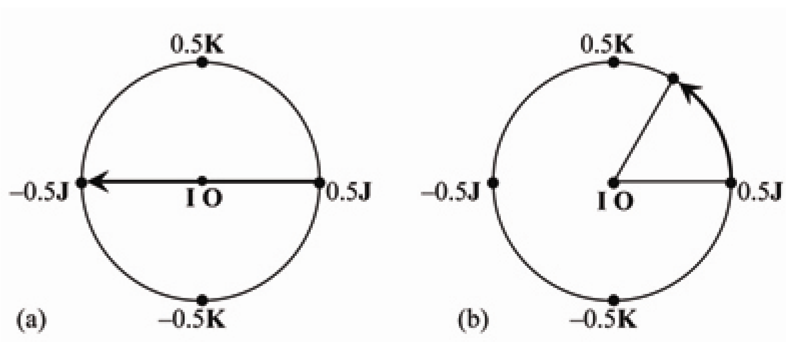

Figure 10 The effect in the antistigmatic plane of the physical operations in Figure 6 on a 0.5-D Jackson cross-cylinder in primary orientation. (a) shows the effect of flipping the lens; the power is changed from 0.5 **J** D to –0.5 **J** diametrically across the circle. (b) shows the effect of turning the lens 30° anticlockwise; the power is changed from 0.5**J** D to 0.25**J** + 0.433 **K** D 60º anticlockwise around the circle.

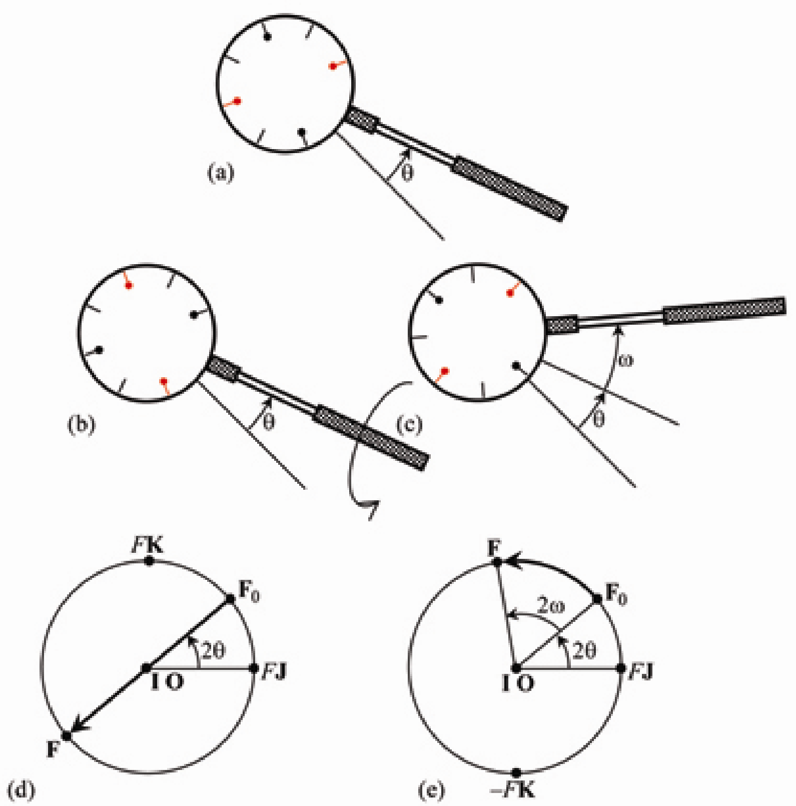

Figure 11 Operations in physical space on a general Jackson cross-cylinder in an arbitrary orientation and the corresponding changes in the abstract space. In (a) the handle is at angle θ anticlockwise from the primary orientation. The lens has power $\mathbf{F}_0$ on the circle in (d) at angle 2θ. Flipping takes the lens from (a) to (b) and the power diametrically across the circle to $\mathbf{F}$ in (d). Turning the lens anticlockwise through angle ω (to (c)) takes the power around the circle anticlockwise through angle 2ω to $\mathbf{F}$ in (e).

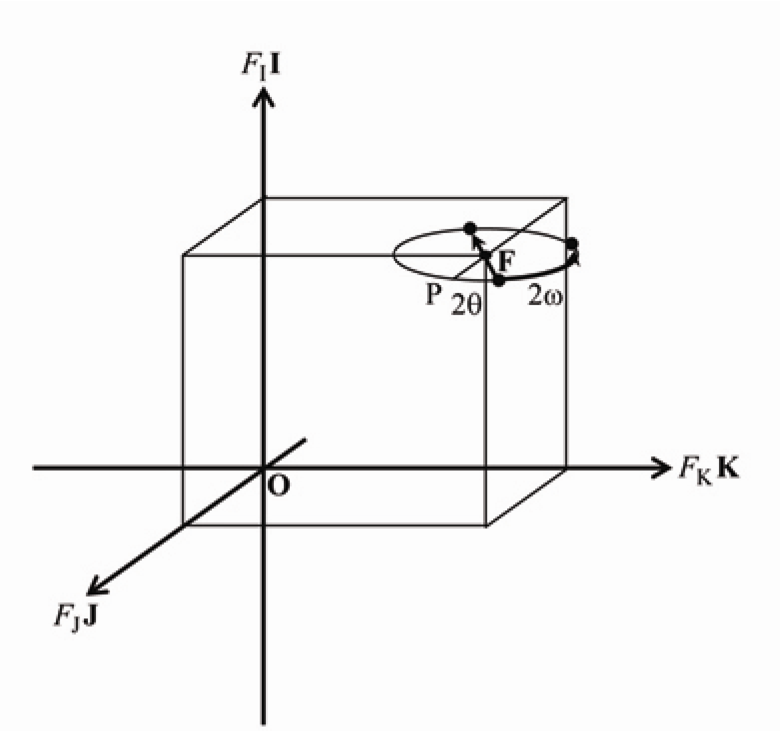

Figure 12 Representation in symmetric dioptric power space of the power of a Jackson cross-cylinder in juxtaposition with a lens of power $\mathbf{F}$. The combination has power represented by the circle; the circle has centre at $\mathbf{F}$ and radius equal to the magnitude of the Jackson cross-cylinder and lies in a plane orthogonal to axis $\mathbf{I}$. The Jackson cross-cylinder is initially at angle θ anticlockwise from the primary orientation. Flipping takes the power of the combination diametrically across the circle. Turning through angle ω in physical space takes the power through angle 2ω in symmetric dioptric power space. The sense of the rotation is the same if viewed down axis $\mathbf{I}$.

Consider a 1-D Jackson cross-cylinder in primary orientation (Figure 7(a)). In symmetric dioptric power space its power is located at **J** D. Let us turn it anticlockwise. A turn of 45° of the actual lens in physical space (to Figure 7(b)) takes the point from **J** D in symmetric dioptric power space to **K** D. All turns take the point along the circle of radius 1 D in Figure 9. Looking down axis **I** (as in Figure 9(b)) the sense of the rotation is the same; that is, an anticlockwise turn in physical space corresponds to an anticlockwise turn in symmetric dioptric power space. The angle, however, is twice as much in symmetric dioptric power space. Thus a physical turn of 45° from (a) to (b) in Figure 7 takes one 90° in symmetric dioptric power space. In general a physical turn of θ produces a turn of 2θ in symmetric dioptric power space. In particular a physical turn of 90º (to (c) in Figure 7) takes one 180° in the space to –**J** D, that is, diametrically across the circle. This is equivalent to the effect of flipping the lens. In other words flipping the lens takes one diametrically across the circle in symmetric dioptric power space. These results are general.

The circles in Figure 9 represent the powers of 0.25-D, 0.5-D, 1-D and 2-D Jackson cross-cylinders. Let us examine the 0.5-D Jackson cross-cylinder in particular. The circle of radius 0.5 D in Figure 10 represents all the powers possible for the lens. The point labelled 0.5**J** D represents the power of the lens when in primary orientation, that is, the orientation in Figure 6(a). The lens in Figure 6(b) has power –0.5**J** D. The operation of flipping the lens in physical space from (a) to (b) in Figure 6 takes one diametrically across the circle in the antistigmatic plane in symmetric dioptric power space as shown in Figure 10(a). The lens in Figure 6(c) has been turned anticlockwise by 30°. The corresponding turn in the plane of antistigmatic powers is twice that, namely 60°, also anticlockwise for the orientation of the space in Figure 10. The resulting power is $\begin{pmatrix} 0.250 & 0.433 \\ 0.433 & -0.250 \end{pmatrix}$ D which is equivalent to $0.250\mathbf{J}+0.433\mathbf{K}$ D. In other words $F_\mathrm{J}=0.250$ D and $F_\mathrm{K}=0.433$ D, in effect the coordinates of the power in the plane of antistigmatic powers.

We now generalize these observations for any Jackson cross-cylinder in any orientation. Figure 11(a) shows an arbitrary Jackson cross-cylinder turned anticlockwise through arbitrary angle θ relative to its primary orientation. Let us call it an F Jackson cross-cylinder. (For example, a 0.5-D Jackson cross-cylinder has $F=0.5$ D.) All the powers possible for the lens are represented by the circle in (d) and in (e). For the orientation in (a) the lens has power $\mathbf{F}_0$; the corresponding point is labelled $\mathbf{F}_0$ in (d) and (e). Notice that it is at angle 2θ. Corresponding to operations on the lens in physical space are changes in the abstract symmetric dioptric power space. In particular flipping, from (a) to (b), takes one diametrically across the circle from $\mathbf{F}_0$ to $\mathbf{F}$ in (d). Turning the lens anticlockwise through angle ω, as shown in (c), takes one anticlockwise through 2ω in the abstract space as shown in (e).

The description in the paragraph above starts with the lens with front surface facing the reader. For completeness we need also to allow the lens in Figure 11(a) to have its back surface facing the observer. Equivalently, this means we need to start with the lens in (b) and apply the two operations to it. Flipping takes it to (a); its power changes from $\mathbf{F}$ to $\mathbf{F}_0$. In the abstract space the change is the reverse of that shown in (d). Turning the lens in (b) through ω takes the power from $\mathbf{F}$ anticlockwise around the circle in the abstract space though 2ω to a point not shown in (e).

Having completed our examination of the Jackson cross-cylinder on its own we now need to examine the Jackson cross-cylinder in combination with other lenses.

The Jackson Cross-Cylinder in Combination with Other Lenses

For thin lenses in juxtaposition the powers add. This is so in particular for a Jackson cross-cylinder in combination with other thin lenses. Suppose there is a lens of power **F** in a trial frame. The power will be a point in symmetric dioptric power space as shown in Figure 12. A Jackson cross-cylinder is now placed against the lens in the trial frame and may be flipped or turned. Everything we have said above about the lens in isolation applies now as well except that the picture is shifted by **F** in symmetric dioptric power space. The juxtaposed lenses have combined power represented by the circle in Figure 12. The circle now is centred on **F** rather than on **O** as was the case for the isolated Jackson Cross-cylinder in Figure 9. It lies in a plane parallel to the antistigmatic plane and orthogonal to axis **I**. For an F Jackson cross-cylinder the radius of the circle is F as before. If the lens is in its primary orientation then the combined power will be at

P in Figure 12. If it has been turned through angle θ in physical space then the point will be displaced through angle 2θ around the circle in the abstract space. Operations on the Jackson cross-cylinder now have the same effect about **F** as shown about **O** in Figure 11: flipping takes one diametrically across the circle and turning takes one around the circle but at twice the rate.

Concluding Remarks

We have examined the first-order optics of the Jackson cross-cylinder in detail. We have looked at relevant optical properties of the lens and we have seen in particular how its power is represented in symmetric dioptric power space. We have seen how the clinical operations of flipping and turning the Jackson cross-cylinder in physical space displace the power in the abstract space. Having examined the particulars of the Jackson cross-cylinder we are now ready to look at the lens as an integral part of the clinical refraction routine which is what we do in a subsequent paper[1].

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Appendix

Consider a thin lens in air of power **F**. Then at a point on the lens with position vector **y** relative to the optical centre the deflectance (prismatic effect) **p** is given by[14-18]

$$\mathbf{p} = -\mathbf{F}\mathbf{y}\ .$$

For a 0.5-D Jackson cross-cylinder in primary orientation

$$\mathbf{p} = -0.5\mathbf{J}\mathbf{y}\ \text{D}$$

or

$$p_1 = -0.5 y_1\ \text{D}$$
$$p_2 = 0.5 y_2\ \text{D}.$$

The sketch in Figure 4(d) is based on these equations.

The thickness at the same point of a thin lens is given approximately by[22]

$$t = t_0 - \frac{\mathbf{y}'\mathbf{F}\mathbf{y}}{2(n-1)}$$

where t_0 is the thickness at the optical centre and n is the index of the lens. In particular for the 0.5-D Jackson cross-cylinder in primary orientation

$$t = t_0 - \frac{y_1^2 - y_2^2}{4(n-1)}\ \text{m}^{-1}\ .$$

Thus a contour of constant thickness t has equation

$$y_1^2 - y_2^2 = 4(n-1)(t_0 - t)\ \text{m}$$

the equation of a thickness contour in Figure 4(c).

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