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Exploded representation of a refracting Surface

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Abstract

The concept of the exploded refracting surface is useful in the optics of contact lenses and vision underwater. The purpose of this paper is to show how to represent a refracting surface as an exploded pair of surfaces separated by a gap of zero width. The analysis is in terms of linear optics and allows for astigmatic and non-coaxial cases.

The concept of the exploded refracting surface is useful in contact lens optics[1, 2] and in the analysis of vision under water, the latter of these being a future goal of the authors. The exploded surface allows one to treat the submerged eye as being in contact with air and so retain its optical character. This allows for an easier comparison of the submerged eye to the eye in air.

The goal of this paper is to show, in general, how to represent a single refracting surface as a system of two juxtaposed single refracting surfaces. Explicit formulae are presented in equations 8 and 9 for the transferences of the two surfaces.

We make use of linear optics[1-15] within which the transference is an important concept. We start by reviewing the transference[9] and more particularly the transference of a single refracting surface to find the general divergence $\mathbf{C}$ and deviation $\boldsymbol{\pi}$ for any single refracting surface. From there we explode the single surface into two surfaces separated by a gap of zero width. The transferences for the two surfaces are found and from this explicit formulae are presented that can be used to calculate the transferences of the two surfaces in the exploded representation of a surface from the information contained in the transference of the original surface. The approach is general and allows for surfaces that may be astigmatic and non-coaxial.

The *augmented ray transference*[9] is

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{e} \\ \mathbf{C} & \mathbf{D} & \boldsymbol{\pi} \\ \mathbf{o}' & \mathbf{o}' & 1 \end{pmatrix}. \qquad (1)$$

The 2×2 sub-matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are respectively called[1] the *dilation*, *disjugacy*, *divergence* and the *divarication*. The 2×1 sub-matrices $\mathbf{e}$ and $\boldsymbol{\pi}$ are called the *translation* and *deviation* respectively and they allow for systems that contain tilted and decentred elements. These are the six fundamental properties of any optical system from which other properties can be derived.[10] $\mathbf{o}$ is the 2×1 null vector and $\mathbf{o}'$ its transpose which together with a scalar 1 make up the bottom row.

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We will often simply refer to **T** as the transference and also often omit the bottom row. Dioptric power **F** is defined by **F** = –**C**.[11]

For a homogeneous gap the transference is[9]

$$T = \begin{bmatrix} \mathbf{I} & \varsigma\mathbf{I} & \mathbf{o} \\ \mathbf{O} & \mathbf{I} & \mathbf{o} \end{bmatrix}, \qquad (2)$$

where ς is the reduced width of the gap. The bottom row is omitted to save space. **I** is the 2 x 2 identity matrix and **O** the 2 x 2 null matrix. A special case is the 5 x 5 identity transference where $\varsigma = 0$.

Of particular relevance here is the transference of a single refracting surface[9]

$$T = \begin{bmatrix} \mathbf{I} & \mathbf{O} & \mathbf{o} \\ -\mathbf{F} & \mathbf{I} & \boldsymbol{\pi} \end{bmatrix} \qquad (3)$$

where **F** is symmetric.[11]

We now examine **F** and $\boldsymbol{\pi}$ in particular.

Dioptric power can be represented as[12]

$$\mathbf{C} = -\mathbf{F} = -\mathbf{K}(n_2 - n_1) \qquad (4)$$

where **K** has been called the curvature matrix[12] of the surface and n_1 and n_2 are the refractive indices of the media upstream and downstream of the surface respectively.

The deviation $\boldsymbol{\pi}$ depends on the non-coaxiality of the surface with respect to the longitudinal axis Z. A surface normal to Z has $\boldsymbol{\pi} = \mathbf{o}$. For $\boldsymbol{\pi} \neq \mathbf{o}$ the surface is tilted by **b** and/or decentered by **c**. In general[13]

$$\boldsymbol{\pi} = (\mathbf{Kc} + \mathbf{b})(n_2 - n_1). \qquad (5)$$

Next we explode T into two surfaces T_1 and T_2, that is, we imagine two surfaces that are separated by a gap of zero width, that have the same curvature as T, and that also have the same amounts of tilt and decentration as T. Upstream of T_1 the medium has refractive index n_1, and downstream of T_2 the medium has refractive index n_2. The medium in the gap between T_1 and T_2 has refractive index n_0. We refer to n_0 as the reference index. n_0 is arbitrary although in most cases it is likely to be 1. From this we know that T_1 has power $\mathbf{F}_1 = \mathbf{K}(n_0 - n_1)$ and deviation $\boldsymbol{\pi}_1 = (\mathbf{Kc} + \mathbf{b})(n_0 - n_1)$. Similarly for T_2 we have $\mathbf{F}_2 = \mathbf{K}(n_2 - n_0)$ and $\boldsymbol{\pi}_2 = (\mathbf{Kc} + \mathbf{b})(n_2 - n_0)$.

Using equation 4 we can represent $\mathbf{F}_1$ as

$$\mathbf{F}_1 = \mathbf{F}\,\frac{n_0 - n_1}{n_2 - n_1}. \qquad (7)$$

and similarly for $\mathbf{F}_2$. Making use of equations 4 and 5 we find similar expressions for $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$. These results lead one to a transference for each surface in terms of **F**, $\boldsymbol{\pi}$ and the respective refractive indices. By adding and subtracting **I** from each of the two transferences, and rearranging the terms, one finds

$$\mathbf{T}_1 = x_1\mathbf{T} + (1 - x_1)\mathbf{I}, \qquad (8)$$

and

$$\mathbf{T}_2 = x_2\mathbf{T} + (1 - x_2)\mathbf{I} \qquad (9)$$

with $x_1 = \dfrac{n_0 - n_1}{n_2 - n_1}$ and $x_2 = \dfrac{n_2 - n_0}{n_2 - n_1}$.

The transferences of two surfaces separated by a gap of zero width is $\mathbf{T}_2\mathbf{I}\mathbf{T}_1$. Therefore, the transference of the system of the two surfaces can simply be represented by $\mathbf{T}_2\mathbf{T}_1$ which recovers the transference **T**.

The result above allows one to represent a single refracting surface as a system of two refracting surfaces, which are separated by gap of zero width that is filled with a medium with a refractive index defined as the reference index. Therefore, given any transference of a single refracting surface T, one can design a system of two juxtaposed refracting surfaces T_1 and T_2, that will yield the same transference.

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