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The concept of the average eye

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Abstract

For most quantitative studies one needs to calculate an average. In the case of refraction an average is readily computed as the arithmetic average of dioptric power matrices. Refraction, however, is only one aspect of the first-order optical character of an eye. The question is: How does one determine an average that represents the average optical character of a set of eyes completely to first order? The exponential-mean-log transference has been proposed recently but it is not without its difficulties. There are four matrices, naturally related to the transference and called the characteristics or characteristic matrices, whose mathematical features suggest that they may provide alternative solutions to the problem of the average eye. Accordingly the purpose of this paper is to propose averages based on these characteristics, to examine their nature and to calculate and compare them in the case of a particular sample of 30 eyes. The eyes may be stigmatic or astigmatic and component elements may be centred or decentred. None turns out to be a perfect average. One of the four averages (that based on one of the two mixed characteristics) is probably of little or no use in the context of eyes. The other three, particularly the point-characteristic average, seem to be potentially useful.

For quantitative research one frequently needs the concept of an average. As part of ongoing work[1, 2] in our research group we have been seeking a way of calculating an average of a set of eyes, an average that represents the complete linear optical character of the eyes. Because the transference embodies the complete first-order optics of the eye it seems reasonable to look for a suitable average eye in terms of transferences. The ordinary arithmetic average, however, must be discarded as a suitable average because it usually violates the symplectic condition and, hence, cannot represent an eye or optical system.[3] Of course an average eye should be an eye that is possible in principle. One possibility, also based on transferences, has been proposed recently but it is not without its difficulties.[3-5] On the other hand there are a number of matrices, related to the transference, which have mathematical features that suggest alternative ways of defining an average eye. The purpose of this paper is to define those other average eyes and briefly examine some of their features. For a randomly-generated set of myopic eyes the different averages are calculated and compared.

If all that matters about an eye is its refraction then calculating an average is an easy matter: one can make use of the dioptric power

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matrix **F** as first pointed out by Keating[6]. If there are N refractions of power $\mathbf{F}_i$ then the average would be

$$\bar{\mathbf{F}} = \frac{1}{N}\sum_{i=1}^{N} \mathbf{F}_i .$$

Although the refraction is clearly an important aspect of the optical nature of an eye it is not the whole story: two eyes with the same refraction may differ in other respects, retinal image sizes for example.

The complete first-order characterization of an optical system, including an eye, requires knowledge of what the system does to any ray traversing it. In linear optics[7-9] one represents the action of the system by means of what we call the ray *transference* **T** of the system. An eye in particular operates on the ray according to the general equation

$$\mathbf{T}\boldsymbol{\gamma}_0 = \boldsymbol{\gamma}.$$

The eye changes the state of the ray from $\boldsymbol{\gamma}_0$ at incidence onto the eye to $\boldsymbol{\gamma}$ at the retina.

With allowance made for astigmatism and prismatic and decentred elements the transference is a 5 x 5 matrix[9] which we can represent as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{e} \\ \mathbf{C} & \mathbf{D} & \boldsymbol{\pi} \\ \mathbf{o}' & \mathbf{o}' & 1 \end{pmatrix} .$$

The entries **A**, **B**, **C**, **D**, **e** and $\boldsymbol{\pi}$ are what we call the six *fundamental* first-order optical properties of the eye.[10] They are themselves matrices, the first four being 2 x 2 and the last two 2 x 1. All other first-order optical properties are called *derived* properties of the eye; they can be calculated from the fundamental properties[11]. One important derived property is the power **F** of the eye; it is defined by[12]

$$\mathbf{F} = -\mathbf{C}.$$

Another is the corneal-plane refraction $\mathbf{F}_0$ of the eye; it is given by[11]

$$\mathbf{F}_0 = \mathbf{B}^{-1}\mathbf{A}. \tag{1}$$

(Notice the dependence of the corneal-plane refraction on the properties **A** and **B** and not directly on the power **F** of the eye.) The bottom row of **T** is trivial; in the interests of conserving space we abbreviate the transference as

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{e} \\ \mathbf{C} & \mathbf{D} & \boldsymbol{\pi} \end{pmatrix}$$

in which the fifth row of four 0s and a 1 is understood.

We can calculate averages for properties (fundamental and derived) each taken separately. Because of the requirement of symplecticity we must not expect those average fundamental properties in combination to constitute a structure that could function together as a whole eye. An eye is, indeed, more than the sum of its parts.[3] Furthermore we should not expect a meaningful average eye to have properties that are averages of those properties of the individual eyes.

A kind of pseudo-geometric average has been proposed recently[3]. It is the *exponential-mean-logarithmic eye*, defined by

$$\bar{\mathbf{T}}_{\mathrm{EML}} = \exp\left(\frac{1}{N}\sum_{i=1}^{N} \log \mathbf{T}_i\right). \tag{2}$$

The exponential-mean-logarithmic eye satisfies the symplectic condition. It seems to be a satisfactory average eye but there are points of criticism[3].

There are five other 5 x 5 matrices, all naturally related to the transference **T**, which might provide satisfactory average eyes[13]. In terms of the fundamental properties they can be expressed as follows:[13]

$$\mathbf{U} = \begin{pmatrix} \mathbf{D}' & -\mathbf{B}' & -\mathbf{D}'\mathbf{e} + \mathbf{B}'\boldsymbol{\pi} \\ -\mathbf{C}' & \mathbf{A}' & \mathbf{C}'\mathbf{e} - \mathbf{A}'\boldsymbol{\pi} \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{B}^{-1}\mathbf{A} & -\mathbf{B}^{-1} & \mathbf{B}^{-1}\mathbf{e} \\ -\mathbf{B}'^{-1} & \mathbf{D}\mathbf{B}^{-1} & -\mathbf{D}\mathbf{B}^{-1}\mathbf{e} + \boldsymbol{\pi} \end{pmatrix}, \tag{3}$$

$$\mathbf{Q} = \begin{pmatrix} \mathbf{C}^{-1}\mathbf{D} & \mathbf{C}^{-1} & \mathbf{C}^{-1}\boldsymbol{\pi} \\ \mathbf{C}'^{-1} & \mathbf{A}\mathbf{C}^{-1} & \mathbf{e} - \mathbf{A}\mathbf{C} - \mathbf{1}\boldsymbol{\pi} \end{pmatrix}, \tag{4}$$

$$\mathbf{M} = \begin{pmatrix} -\mathbf{D}^{-1}\mathbf{C} & \mathbf{D}^{-1} & -\mathbf{D}^{-1}\boldsymbol{\pi} \\ \mathbf{D}'^{-1} & \mathbf{B}\mathbf{D}^{-1} & \mathbf{e} - \mathbf{B}\mathbf{D}^{-1}\boldsymbol{\pi} \end{pmatrix}, \tag{5}$$

$$\mathbf{N} = \begin{pmatrix} -\mathbf{A}^{-1}\mathbf{B} & \mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{e} \\ \mathbf{A}'^{-1} & \mathbf{CA}^{-1} & -\mathbf{CA}^{-1}\mathbf{e} + \boldsymbol{\pi} \end{pmatrix}. \quad (6)$$

In each matrix the fifth row is trivial and understood. Formulae for determining the transference **T** from each of these matrices are given elsewhere.[13]

Matrix **U** is the inverse of **T**. Its mathematical character is essentially the same as that of **T**. One can define an exponential-mean-logarithmic average $\overline{\mathbf{U}}_{\mathrm{EML}}$ by replacing **T** by **U** in equation 2. However, as might be expected, the average eye defined that way is identical to the average eye defined by $\overline{\mathbf{T}}_{\mathrm{EML}}$. Otherwise the matrix **U** is of no interest and will not be considered further.

We call **P**, **Q**, **M** and **N** the *characteristic* matrices of the system[13]. **P** is the *point* characteristic matrix, **Q** the *angle* characteristic matrix and **M** and **N** are what we call the *first* and *second mixed* characteristic matrices respectively. The inverses that appear in equations 3 to 6 are potential sources of difficulty which we shall examine below.

The characteristic matrices exhibit a new and suggestive mathematical feature: as inspection confirms the top-left 4 x 4 submatrix in each case is symmetric, and the whole matrix has 10 independent entries. Together with the last column there are 14 independent entries. This means one can calculate an arithmetic average of any of the four characteristic matrices in the usual way and that the average is meaningful in the sense that the average is a possible eye. One does not have to worry about symplecticity because, in effect, it is accounted for in the symmetric 4 x 4 submatrix. We look separately a little further at each of these average eyes.

We can define an average eye by means of the ordinary arithmetic average of the point characteristic matrix **P**:

$$\overline{\mathbf{P}} = \frac{1}{N}\sum_{i=1}^{N}\mathbf{P}_i\,. \quad (7)$$

From the transference for each eye we calculate $\mathbf{P}_i$ for the eye according to equation 3. We obtain the average $\overline{\mathbf{P}}$ using equation 7. Finally the transference $\overline{\mathbf{T}}_{\mathbf{P}}$ of the average eye can be calculated from $\overline{\mathbf{P}}$ using the equation given elsewhere[13]. This is the *point-characteristic average eye*. It is evident from equations 1 and 3 that the refraction of a point-characteristic average eye is the same as the average of the refractions of the eyes, which makes this particular average eye of potential interest in optometry.

Submatrix **B** appears as an inverse in **P** (equation 3). Thus a point-characteristic average eye does not exist if at least one of the eyes in the sample has a singular **B**. In fact one would anticipate difficulties if any eye in the set had a **B** that approached singularity. There seems to be no problem in practice, however; all 'reasonable' eyes have **B**s that are far from singular. (A thin or thin-equivalent system has singular **B**.)

Instead of point characteristic **P** we can obtain an average by averaging the angle-characteristic **Q** to obtain the average $\overline{\mathbf{Q}}$. *The angle-characteristic average eye*, with transference $\overline{\mathbf{T}}_{\mathbf{Q}}$, seems satisfactory provided that (as one sees from equation 4) the divergence **C** of none of the eyes is singular or near singular. For ordinary eyes there seems to be no difficulty.

In exactly the same way we can use the average first mixed characteristic $\overline{\mathbf{M}}$ to define a *first mixed-characteristic average eye* with transference $\overline{\mathbf{T}}_{\mathbf{M}}$. Here singular or near-singular property **D** (equation 5) is a potential problem although it is an unlikely problem in the case of all but extreme eyes.

Lastly one can contemplate a *second mixed-characteristic average eye* with transference $\overline{\mathbf{T}}_{\mathbf{N}}$ based on the arithmetic average $\overline{\mathbf{N}}$. This average does not exist when one eye in the set has singular **A** (equation 6) and may be problematic when an eye has near-singular **A**. Thus the set cannot include an eye that is emmetropic (**A** = **O**) or near emmetropic. Because of the importance of eyes that are emmetropic or near emmetropic it seems unlikely that this type of average would be of any interest.

We are left with three potentially useful types of average eye based on characteristic matrices, namely those based on averages of **P**, **Q** and **M**. However each contains the inverse of a matrix and so is potentially problematic even though

Table 1 The ray transferences of the first five in a set of 30 randomly-generated myopic eyes. The corneal-plane refractions are given for interest. The refracting elements have been assumed all to be centred on the common longitudinal axis; the fifth row and column of each transference are trivial and omitted. The units are m for the top-right 2 x 2 submatrix and D for the bottom-left 2 x 2 submatrix.

Transference	Corneal-plane refraction
$\mathbf{T}_1 = \begin{pmatrix} -0.2066 & -0.0031 & 0.0200 & 0.0000 \\ -0.0031 & -0.2240 & 0.0000 & 0.0200 \\ -58.8160 & -0.0853 & 0.8569 & 0.0017 \\ -0.0841 & -59.5090 & 0.0017 & 0.8588 \end{pmatrix}$	−10.30 −0.90 x 171
$\mathbf{T}_2 = \begin{pmatrix} -0.1641 & 0.0060 & 0.0197 & 0.0000 \\ 0.0060 & -0.1399 & 0.0000 & 0.0197 \\ -57.9190 & 0.3455 & 0.8670 & 0.0024 \\ 0.3415 & -56.9734 & 0.0024 & 0.8637 \end{pmatrix}$	−7.03 −1.36 x 76
$\mathbf{T}_3 = \begin{pmatrix} -0.1658 & 0.0022 & 0.0199 & 0.0000 \\ 0.0022 & -0.1389 & 0.0000 & 0.0199 \\ -57.4291 & 0.1066 & 0.8636 & 0.0004 \\ 0.1059 & -56.3178 & 0.0004 & 0.8625 \end{pmatrix}$	−6.98 −1.36 x 85
$\mathbf{T}_4 = \begin{pmatrix} -0.0888 & 0.0129 & 0.0196 & 0.0000 \\ 0.0130 & -0.1036 & 0.0000 & 0.0197 \\ -54.9406 & 0.6069 & 0.8659 & 0.0013 \\ 0.6107 & -55.4450 & 0.0013 & 0.8701 \end{pmatrix}$	−4.14 −1.52 x 30
$\mathbf{T}_5 = \begin{pmatrix} -0.1734 & 0.0064 & 0.0197 & 0.0000 \\ 0.0064 & -0.1598 & 0.0000 & 0.0197 \\ -58.2353 & 0.3021 & 0.8632 & 0.0006 \\ 0.3016 & -57.6426 & 0.0006 & 0.8629 \end{pmatrix}$	−7.96 −0.95 x 68

one would expect difficulties only with highly unrealistic types of eyes. The exponential-mean-logarithmic average eye does not involve potentially non-existent inverses. However this average is nevertheless potentially problematic in highly unusual circumstances.[3]

Implicit in what has been discussed is the fact that all real matrices are realizable as optical systems[14, 15]. Therefore, provided the average transferences exist and are real, the average eyes defined here do exist and are meaningful.

Table 1 lists the transferences of the first five of a set of 30 myopic eyes generated using random numbers and the Gullstrand simplified eye as a starting point. The various separating distances in the eye and the scalar component F_I and the ortho- F_J and oblique F_K antistigmatic components of the dioptric power of each of the four refracting surfaces were selected randomly between pre-selected limits. The sphero-cylindrical corneal-plane refractions, calculated by means of equation 1, are listed for interest in Table 1. The five different average eyes described here were computed and their transfer-

Table 2 The transferences of five different types of average eye calculated for the sample of 30 eyes partially listed in Table 1. The fifth row and fifth column of each transference are trivial and omitted. The corneal - plane refraction is given in each case for interest.

Transference	Corneal-plane refraction
$\bar{\mathbf{T}}_{\text{EML}} = \begin{pmatrix} -0.1444 & 0.0009 & 0.0198 & 0.0000 \\ 0.0009 & -0.1451 & 0.0000 & 0.0198 \\ -56.7045 & 0.0725 & 0.8655 & 0.0011 \\ 0.0724 & -56.7361 & 0.0011 & 0.8655 \end{pmatrix}$	−7.24 −0.11 x 35
$\bar{\mathbf{T}}_{\mathbf{P}} = \begin{pmatrix} -0.1436 & 0.0009 & 0.0198 & 0.0000 \\ 0.0009 & -0.1443 & 0.0000 & 0.0198 \\ -56.6900 & 0.0733 & 0.8655 & 0.0011 \\ 0.0733 & -56.7251 & 0.0011 & 0.8655 \end{pmatrix}$	−7.20 −0.11 x 35
$\bar{\mathbf{T}}_{\mathbf{Q}} = \begin{pmatrix} -0.1429 & 0.0010 & 0.0198 & 0.0000 \\ 0.0010 & -0.1440 & 0.0000 & 0.0198 \\ -56.6422 & 0.0742 & 0.8655 & 0.0011 \\ 0.0744 & -56.6924 & 0.0011 & 0.8655 \end{pmatrix}$	−7.17 −0.12 x 31
$\bar{\mathbf{T}}_{\mathbf{M}} = \begin{pmatrix} -0.1439 & 0.0009 & 0.0198 & 0.0000 \\ 0.0009 & -0.1447 & 0.0000 & 0.0198 \\ -56.6898 & 0.0733 & 0.8654 & 0.0011 \\ 0.0734 & -56.7244 & 0.0011 & 0.8654 \end{pmatrix}$	−7.22 −0.11 x 35
$\bar{\mathbf{T}}_{\mathbf{N}} = \begin{pmatrix} -0.1281 & 0.0006 & 0.0198 & 0.0000 \\ 0.0006 & -0.1328 & 0.0000 & 0.0198 \\ -56.1711 & 0.0622 & 0.8652 & 0.0012 \\ 0.0642 & -56.3787 & 0.0011 & 0.8649 \end{pmatrix}$	−6.47 −0.25 x 8

ences are listed in Table 2. Again the corneal-plane refractions are given for interest. All five averages exist and all represent possible eyes. The averages are similar although the second mixed-characteristic average eye (based on characteristic **N**) is somewhat different from the rest. This observation supports the statement above that the second mixed-characteristic average was unlikely to be of interest in the context of eyes. If one were interested only in the refractions one would say that the averages, with the exception of that based on **N**, do not differ clinically.

In conclusion, then, we see that an eye is more than the sum of its parts. Averages taken separately of individual components of eyes do not in general result in an average eye that represents the complete first order optical character of the eyes. In other words such an average is not meaningful. The five average eyes defined here are usually meaningful and each one may conceivably have its use in particular contexts. The second mixed-characteristic average, that based on **N**, is probably of little interest in the case of eyes. The point characteristic average eye, that based on **P**, is of particular interest because it is the only average whose refraction is the average of the refractions in the set of eyes being averaged.

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