CONFERENCE PAPER

Surfaces of tilted powers in symmetric dioptric power space*

Alan Rubin and WF Harris+**

Optometric Science Research Group, Department of Optometry, University of Johannesburg, PO Box 524, Auckland Park, 2006 South Africa

**<arubin@uj.ac.za>
+<wfharris@uj.ac.za>

Tilting a thin lens of power in general changes the effective power of the lens to (see equations 1 to 4). In this paper lenses of uniform refractive index n are regarded as being in media of index n^*. For instance, a spectacle lens in air has $n^*=1$ whereas an intraocular contact lens or implant *in situ* has $n^* > 1$. For a given power, should $n^* > 1$ the effects of tilts will be greater than for a comparable spectacle lens in air and thus care needs to be exercised to minimize tilting for intraocular contact lens or implants. This paper uses an original mathematical[1-2] and graphical approach to demonstrate these effects for both stigmatic[2] (see Figure 1) and astigmatic (Figures 2 and 3) powers. If a lens is tilted through tilt angle ϕ about a tilt axis θ, a surface of effective, or tilted, powers results and such surfaces can be represented in symmetric dioptric power space. This approach greatly assists in simplifying and understanding the complicated nature and extent of the influences of the tilting process as applied to the untilted lens of given power. The method is general, and can be used with any tilt including the special cases of faceform or pantoscopic tilt that are commonly encountered in optometry and ophthalmology. Underlying mathematical issues are described by Harris[1-2], Blendowske[3] and Keating[4, 5], but much of the emphasis in this paper will be placed directly on the interpretation and meaning of the surfaces themselves. This paper will clearly illustrate how modern multivariate approaches to dioptric power can make difficult optical issues much more intuitively, and intellectually, accessible.

The following equations, from Harris[1-2], are used to generate the tilted surfaces in symmetric dioptric power space:

$$\mathbf{F} = \mathbf{\Phi}\mathbf{F}_0\mathbf{\Phi} \qquad (1)$$

where $\mathbf{F}$ is the tilted power, $\mathbf{F}_0$ is the untilted power and $\mathbf{\Phi}$ is the general tilt matrix given by

$$\mathbf{\Phi} = \mathbf{R}_\theta \mathbf{\Phi}_\theta \mathbf{R}'_\theta \qquad (2)$$

where

$$\mathbf{R}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad (3)$$

is the rotation matrix and

*Based on a paper presented at Mopane 2006A, an International Conference on Visual Optics held at Mopani Camp, Kruger National Park, Limpopo Province, South Africa, August 6 to 9, 2006.
Recieved 2 February 2007; revised version accepted 13 February 2007.

$$\Phi_0 = \sqrt{1+\frac{n^*}{2n}\sin^2\phi}\begin{pmatrix}1 & 0\\ 0 & \sec\phi\end{pmatrix} \qquad (4)$$

is the pantoscopic tilt matrix.

Figure 1 demonstrates in symmetric dioptric power space the surface of tilted powers for a lens having (untilted) scalar power of $\mathbf{F}_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ D.

This case has been described in detail elsewhere.[2] The origin has been shifted so that it lies at the untilted power $\mathbf{F}_0$. The surface of tilted powers is roughly conical. For a given tilt axis θ tilting from $\phi = 0^0$ defines a generator of the cone. Circles of constant tilt angle are shown for ϕ equal to 5^0, 10^0, 15^0, 20^0 and 25^0. Increasing the tilt increases the stigmatic component of the power. It also takes one away from the stigmatic axis; in other words the power becomes increasingly astigmatic.

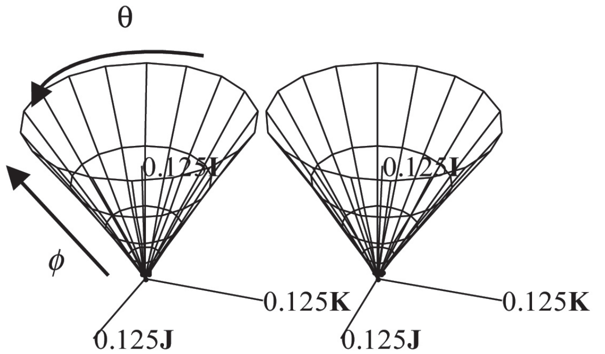

Figure 1: The surface of tilted powers in symmetric dioptric power space for a thin lens of scalar power

I D or $\mathbf{F}_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, or 1 D in conventional terms,

is represented by means of a stereo-pair. Readers should fixate on a point in front of the plane of the paper to see the 3-dimensional percept. The lens, in air (n^* =1), has refractive index 1.5. The surface is described as a pseudo-cone[2] and here consists of six closed curves (or circles) of constant tilt angle ϕ as well as curves (the generators of the pseudo-cone) of constant tilt axis θ. In this surface, for each of the closed curves the tilt axis increases from 0 to 180^0 in 10^0 intervals (ϕ constant) while for each generator, the tilt angle increases from 0 to 25° in 5° steps (θ constant). The origin is the untilted power $\mathbf{F}_0$ and the axis length is 0.125 D.

In Figure 2, the surface of tilted powers for a pure cylinder, or a singular power matrix, is represented and this surface is very different from that for the scalar power (Figure 1). Again the origin represents the untilted power and the same increments in tilt axis, tilt angle and scale are used as for Figure 1. The surface appears flat but in actuality is slightly curved, especially near the origin (a complete description of its geometry remains for further discussion in a subsequent paper).

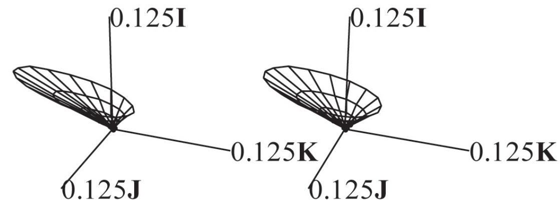

Figure 2: The surface of tilted powers in symmetric dioptric power space for a thin lens of singular power $\mathbf{F}_0 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ D, or cylinder 0 1 x 90 in conventional notation. The lens, a pure cylinder in air (n^* =1), has refractive index 1.5. The surface consists of curves of constant tilt angle ø and curves of constant tilt axis θ with the origin being the untilted power $\mathbf{F}_0$. Intervals for tilt axes and for tilt angles, and the axis length are all the same as for Figure 1.

Another type of thin lens is the Jackson Crossed Cylinder (JCC) with antistigmatic power. The surface of tilted powers for such a thin lens is illustrated in Figure 3. Unlike in Figures 1 and 2, here θ and ϕ were both incremented in smaller intervals, namely [0:2.5:180)° and [0:5:55)° respectively. The scale is an eighth that of that of Figures 1 and 2.

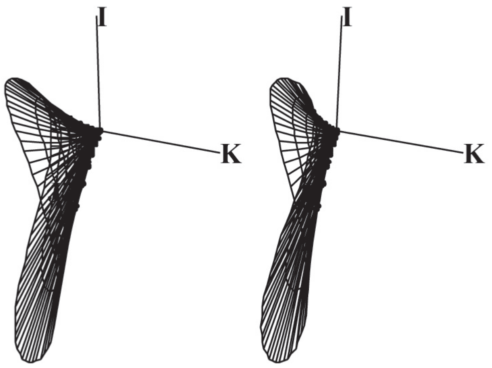

Figure 3: The surface of tilted powers in symmetric dioptric power space for a thin lens of antistigmatic power

$\mathbf{F}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ D, or −1 2 × 90 in conventional terms.

The lens is in air (n^* =1) and has refractive index 1.5. The surface is self-intersecting and the origin is the untilted power $\mathbf{F}_0$. The scale is an eighth of that used in Figures 1 and 2.

Surfaces of tilted power for seven thin lenses are shown in Figure 4, which also includes those for Figures 1 to 3. In contrast to Figures 1 to 3 the surfaces are now shown in their proper positioning in symmetric dioptric power space. From the top in the figure the surfaces are for untilted powers

$$\begin{pmatrix} 1 & 0 \\ 0 & 1.5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -0.5 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & -1.5 \end{pmatrix}, \text{D}$$

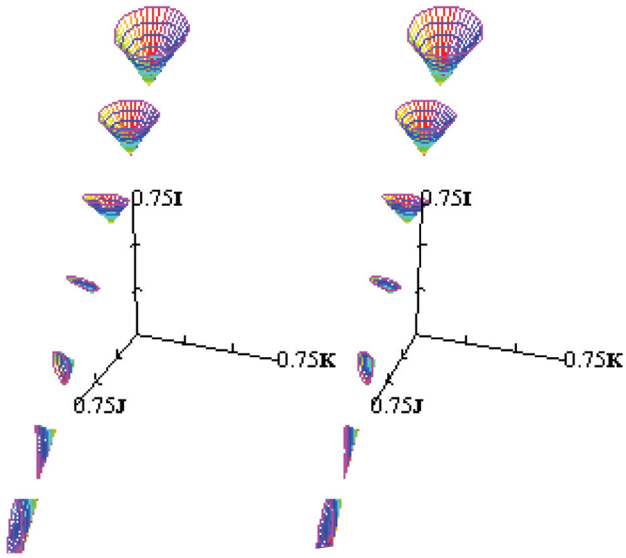

Figure 4: Surfaces of tilted power in symmetric dioptric power space for thin lenses of stigmatic and astigmatic types. All lenses, in air (n^* =1), have refractive index 1.5 and the origin is null power. For all surfaces, θ and ϕ are incremented in small intervals, namely $[0:2.5:180)^\circ$ and $[0:5:60)^\circ$ respectively.

Similar surfaces of tilted power can be represented in symmetric dioptric power space for thin lenses of any power. For tilt angles ϕ larger than about 30° the surfaces probably lose accuracy significantly.[5]

Acknowledgement

WF Harris acknowledges support from the National Research Foundation under Grant Number 2053699.

References

1. WF Harris. Tilted power of thin lenses. *Optom Vis Sci* 2002 **79** 512-515.
2. WF Harris. The pseudo-conical surface of tilted powers of a stigmatic thin lens. *S Afr Optom* 2006 **65** 85-91.
3. Blendowske R. Oblique central refraction in tilted spherocylindrical lenses. *Optom Vis Sci* 2002 **79** 68-73.
4. Keating MP. Oblique central refraction in spherocylindrical lenses tilted around an off-axis meridian. *Optom Vis Sci* 1993 *70* 785-791.
5. Keating MP. Oblique central refraction in spherocylindrical corrections with both faceform and pantoscopic tilt. *Optom Vis Sci* 2002 **79** 68-73.