Ray pencils of general divergency

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Abstract

That a thin refracting element can have a dioptric power which is asymmetric immediately raises questions at the fundamentals of linear optics. In optometry the important concept of vergence, in particular, depends on the concept of a pencil of rays which in turn depends on the existence of a focus. But systems that contain refracting elements of asymmetric power may have no focus at all. Thus the existence of thin systems with asymmetric power forces one to go back to basics and redevelop a linear optics from scratch that is sufficiently general to be able to accommodate such systems. This paper offers an axiomatic approach to such a generalized linear optics. The paper makes use of two axioms: (i) a ray in a homogeneous medium is a segment of a straight line, and (ii) at an interface between two homogeneous media a ray refracts according to Snell's equation. The familiar paraxial assumption of linear optics is also made. From the axioms a pencil of rays at a transverse plane T in a homogeneous medium is defined formally (Definition 1) as an equivalence relation with no necessary association with a focus. At T the reduced inclination of a ray in a pencil is an affine function of its transverse position. If the pencil is centred the function is linear. The multiplying factor $\mathbf{M}$, called the *divergency* of the pencil at T, is a real 2×2 matrix. Equations are derived for the change of divergency across thin systems and homogeneous gaps. Although divergency is undefined at refracting surfaces and focal planes the pencil of rays is defined at every transverse plane in a system (Definition 2). The eigenstructure gives a principal meridional representation of divergency; and divergency can be decomposed into four natural components. Depending on its divergency a pencil in a homogeneous gap may have exactly one point focus, one line focus, two line foci or no foci. Equations are presented for the position of a focus and of its orientation in the case of a line focus. All possible cases are examined. The equations allow matrix step-along procedures for optical systems in general including those with elements that have asymmetric power. The negative of the divergency is the (generalized) vergence of the pencil.

Key words: asymmetric dioptric power, vergence, step-along procedures, focal lines, divergency, divergence

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Introduction

The concept of vergence lies at the heart of much optometric thought and practice. It is in terms of vergence that much of the optics is understood. That includes the concepts of dioptric power and refractive compensation, perhaps the most important measurement that the optometrist makes. It also includes related concepts such as back-vertex power and effective power or effectivity, commonly used to convert between spectacle-plane and corneal-plane refractive compensation. Typically vergence is defined in terms of the reciprocal of the distance to a focal point.[1-7] But how does one define vergence if there is no focus at all? This question arises naturally in the light of a recent paper[8] which describes a thin lens, not plano in power, which has no focus anywhere. That such a lens becomes a possibility immediately throws into question the very foundations of the optics employed in optometry and ophthalmology. Even the concept of a pencil of rays becomes problematic if there is no focus.

Vergence is commonly thought of in terms of curvature of a wavefront. But is it meaningful to talk of wavefronts if there is no focus and one cannot be sure of the meaning of pencils? What of step-along calculational schemes that involve vergence? Do the formulae on which they are based remain valid for an optical system with an element that has power of this new type? In order to accommodate this type of lens in our optical thinking we have no choice but to go back to the fundamentals and develop a whole new approach.

The objective here is, first, to propose a general definition for a pencil of light which does not involve a focus and, then, to examine its behaviour through optical systems in general. The approach is axiomatic. This paper develops the mathematics; an accompanying paper[9] illustrates pencils and foci graphically.

We shall accept only the following:

AXIOM 1 A ray in a homogeneous medium is a segment of a straight line.

AXIOM 2 At an interface between two homogeneous media a ray refracts according to Snell's equation.

We are conscious of the fact that we are working with a very simple model for optics and that applications are likely to be limited. Nevertheless we believe the model is worth exploring for its own sake. No doubt we are pushing the limits for conventional applications but just where those limits are will be for future research to clarify. Our approach is exclusively in terms of rays. In particular we disregard wavefronts entirely. Mathematically we follow the usual approach of Gaussian and linear optics in treating only paraxial behaviour. This allows us to make the usual small-angle assumption and so avoid trigonometric functions.

We begin with a brief review of the concept of vergence as used in familiar Gaussian optics, the two-dimensional linear optics associated with scalar powers. We then review the generalization of these basics to conventional three-dimensional linear optics which allows for astigmatism. These preliminaries lead to a proposal for a definition, Definition 1, for a pencil of rays at a transverse plane in a homogeneous medium that is independent of the notion of a focus. An integral part of the definition is a quantitative measure of the state of the pencil, its divergency. There are singular points (point and line foci) where divergency is not defined. Definition 2 extends the definition of a pencil to all points in an optical system, including focal planes and refracting surfaces. Of course such definitions make sense only if they turn out to be at least as useful as the conventional approach. We show that these definitions provide an approach of which the conventional treatment is a special case. In other words our approach allows one to do everything the conventional approach does, and much more; it allows one to handle the optics of systems containing lenses of the new type as well. It represents a generalization of conventional linear optics. We show this by deriving basic equations that represent the behaviour of the pencil and its divergency, first through a thin system and then across a homogeneous gap. We see how to perform step-along calculations in optical systems that may contain lenses of the new type. The eigenstructure of the divergency defines the principal divergency or divergencies, if any, and the corresponding principal meridians of the pencil. We examine the principal meridians and the conditions under which one obtains point or line foci or no foci. (As we shall see it turns out to be possible for there to be no foci at all whether they be real or virtual in optometric or ophthalmological terms.) Finally we show how to calculate the location of foci, when they occur, and the orientation of line foci, when they occur.

The focus of this paper is on rays. Although the pencils of the type described here can result from thin lenses of the sort described before[8] (necessarily rough and Fresnel-like and, hence, of limited optical quality) other possibilities are not excluded. The paper, then, does not assume the existence of thin systems of the particular type described before. However it is general enough to be able to accommodate them.

Vergence in Gaussian optics

In Gaussian optics vergence is defined in terms of wavefronts and focal points.[1-7] Figure 1(a) shows a pencil of rays diverging from a point focus F. The medium is uniform and has index of refraction n. W represents a diverging wavefront; it is an arc of a circle with centre at F and radius l. For diverging wavefronts, however, l is assigned a negative value. For converging wavefronts the radius is positive. Alternatively one thinks of measuring l from W to F; if the direction from W to F matches the direction in which the light is traveling then $l > 0$ m; if (as in the figure) the direction from W to F is opposite to the direction of the light then $l < 0$ m. At W the pencil has *reduced vergence* defined by

$$L := \frac{n}{l} \cdot \tag{1}$$

Usually we shall drop the qualifier *reduced* and refer simply to the *vergence*.

The usual approach is to think of the wavefront as travelling through the optical system. Its vergence changes as it does so. There are two fundamental processes: change across a thin system and change across a homogeneous gap. As the wavefront crosses a thin system (a refracting surface or thin lens for example) of dioptric power F its vergence changes from L_0, immediately before the system, to L_1, immediately after it. The relationship is

$$L_0 + F = L_1 \ . \tag{2}$$

As the wavefront crosses a homogeneous gap of width z its vergence changes from L_1 to L_2 according to

$$\frac{L_1}{1-\zeta L_1} = L_2 \tag{3}$$

where

$$\zeta := \frac{z}{n} \tag{4}$$

is the reduced width of the gap.

The behaviour of light through an optical system is analyzed by successive application of Equations 2 and 3 in the step-along procedure. (For a discussion of Equations 1 to 4, their use and generalizations the reader is referred to an earlier paper[10] and the references cited therein.)

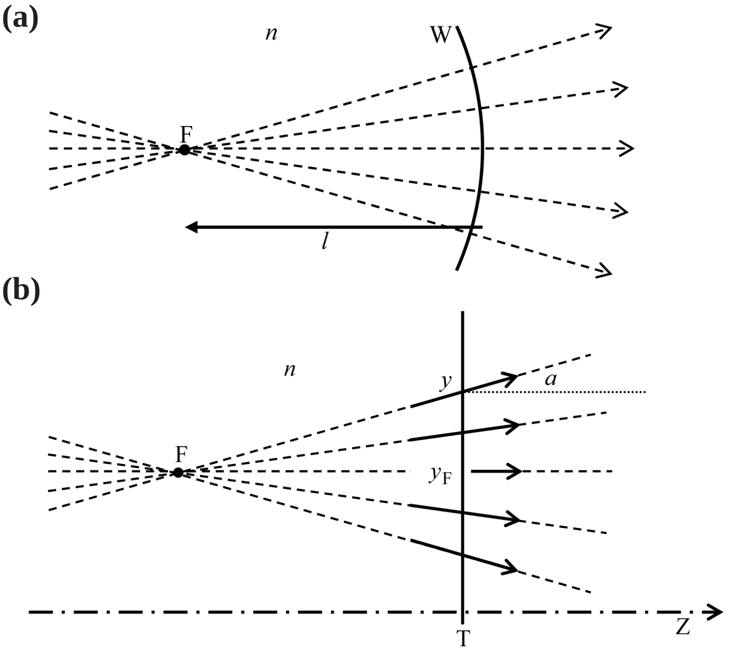

Figure 1 Vergence defined in terms of a wavefront W (a) and in terms of rays (b).

Vergence in conventional linear optics

In conventional linear optics Equations 2 and 3 generalize as follows:[10-17]

$$\mathbf{L}_0 + \mathbf{F} = \mathbf{L}_1 \tag{5}$$

and

$$\mathbf{L}_1(\mathbf{I} - \zeta\mathbf{L}_1)^{-1} = \mathbf{L}_2 \ . \tag{6}$$

All the bold-face upper-case letters represent symmetric 2×2 matrices. **F** is the dioptric power matrix of Fick[18] and Long[19] and **L** is the vergence matrix of Fick[11] and Keating[17, 20]. What matters for us here, however, is that these matrices all have to be symmetric. Because they are symmetric the principal meridians of power and of vergence are orthogonal if the lens or vergence is astigmatic. One obtains a point focus if the lens is not astigmatic and a pair of axially-separated and orthogonal line foci if it is astigmatic.

What happens, now, if the optical system in question contains a component thin system whose power **F** is asymmetric? (Thin systems with asymmetric powers are described elsewhere[8].) Do Equations 5 and 6 still hold? If Equation 5 does hold then the implication would seem to be that wavefronts can have vergence **L** that is asymmetric. Is that meaningful? What is the geometry of a wavefront with asymmetric vergence? What is meant by a pencil of rays with such a wavefront? We can take very little for granted and have to begin again with the fundamentals. To motivate the definition we are going to make below we first need to revisit the concept of vergence but in terms of rays rather than wavefronts. Wavefronts will, in fact, play no role in the development.

A ray interpretation of vergence in Gaussian optics

Figure 1(b) shows the same situation as in (a) except that, instead of the wavefront, there are short segments of rays crossing a transverse plane T. Z is a longitudinal axis. Consider any ray in the pencil. For example, let us choose the top ray shown at T. It has inclination *a* relative to Z and it intersects T in a point with transverse position *y*. Let the focus F be located at a transverse position y_{F} relative to Z. It follows from Figure 1(b) that

$$\tan a = \frac{y - y_{\mathrm{F}}}{-l} \ . \tag{7}$$

The minus sign is required in the denominator because $l < 0$. In Gaussian optics one makes the approximation that *a* is close to zero. Provided *a* is in radians this allows one to write Equation 7 as

$$na = -\frac{n}{l}y - \left(-\frac{n}{l}\right)y_{\mathrm{F}} \, . \tag{8}$$

Applying Equation 1 we obtain

$$\alpha = -Ly - (-L)y_{\mathrm{F}} \tag{9}$$

where

$$\alpha = na \tag{10}$$

is the reduced inclination of the ray. Let us rewrite this equation as

$$\alpha = -L\,y + \alpha^0 \ . \tag{11}$$

This tells us that α^0 is the reduced inclination of a ray in the pencil that happens to intersect transverse plane T at the longitudinal axis Z (where $y = 0$ m). In Figure 1(b) in particular $\alpha^0 < 0$; the corresponding ray is not shown.

Equation 11 shows that, in a pencil of rays, the relationship between the position and reduced inclination of the rays is that of a straight line. The vergence of the pencil is the negative of the slope of the straight line.

Equation 11 also suggests that one should replace vergence by its negative. Accordingly we make the definition made before[21], namely

$$M := -L \ . \tag{12}$$

M is what we call the *divergency* of the pencil at transverse plane T. Thus the rays in the pencil obey

$$\alpha = M\,y + \alpha^0 \, . \tag{13}$$

where *M* and α^0 are two constants that characterize the pencil at the transverse plane. (*Divergency* is distinct from *divergence*. The former is a property of light, the latter a property of a lens or other optical system.[21] We shall meet divergence below.)

Technically the relationship represented by Equation 13 is called *affine*.[22] When the constant α^0 is zero it is called *linear*.

Equation 13 shows how the properties of the rays in a pencil are related in a transverse plane. At least in principle we could, if given the positions and inclinations of rays in a set of rays, determine whether they belonged to the pencil or not. The equation, therefore, provides us with a definition of a pencil based on local properties of the rays in it rather than on some remote focal point. We can say that a set of rays in a transverse plane is a pencil if the reduced inclination of a ray is an affine function of its transverse position.

A general definition of a pencil of rays at a transverse plane

Equation 13 applies in the plane of the paper. However it suggests a natural generalization in three dimensions:

$$\boldsymbol{\alpha} = \mathbf{My} + \boldsymbol{\alpha}^0. \tag{14}$$

Consider a set of rays intersecting a transverse plane T in a homogeneous medium. Two of the rays are shown in Figure 2. By Axiom 1 they are straight lines. Relative to longitudinal axis Z a ray has reduced inclination $\boldsymbol{\alpha}$ and transverse position $\mathbf{y}$. $\boldsymbol{\alpha}$ is a vectorial angle; like $\mathbf{y}$ it has two rectangular components and is represented as a 2×1 matrix.

In the light of the discussion so far we are moved to make the following definition of a pencil of rays at a transverse plane in a homogeneous medium:

DEFINITION 1 Consider a set $\mathcal{P}$ of rays in a homogeneous medium. Suppose that, in a transverse plane T, a ray in $\mathcal{P}$ has transverse position $\mathbf{y}$ and reduced inclination $\boldsymbol{\alpha}$ relative to a longitudinal axis Z. If there exists a particular real 2×2 matrix $\mathbf{M}$ and a particular real 2×1 matrix $\boldsymbol{\alpha}^0$ such that $\boldsymbol{\alpha} = \mathbf{My} + \boldsymbol{\alpha}^0$ for every ray in $\mathcal{P}$ then $\mathcal{P}$ is called a *pencil at transverse plane* T and $\mathbf{M}$ is called the divergency of the pencil at T. $\boldsymbol{\alpha}^0$ is the reduced inclination of a ray that intersects T in Z. It has no units. $\mathbf{M}$ has the units of reciprocal length, usually dioptres D.

An abbreviated form of Definition 1 might read as follows: a set of rays in a homogeneous medium is a pencil at a transverse plane if the reduced inclination of every ray is an affine function of its transverse position.

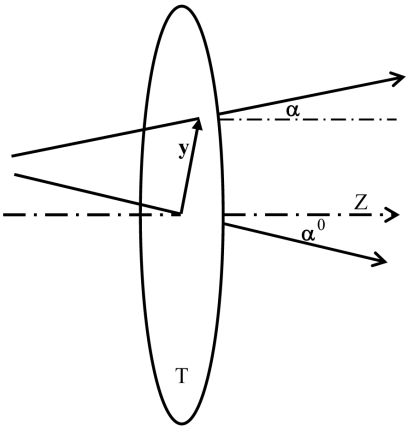

Figure 2 A set of rays intersecting a transverse plane T. Only two of the rays are shown explicitly. One has position vector $\mathbf{y}$ and reduced inclination $\boldsymbol{\alpha}$, both being relative to the longitudinal axis Z. The other ray intersects T in Z and has reduced inclination $\boldsymbol{\alpha}^0$. If there exists a matrix $\mathbf{M}$ such that Equation 14 holds for every ray in the set then the set is a pencil at T and $\mathbf{M}$ is its divergency there.

If $\boldsymbol{\alpha}^0 = \mathbf{o}$, where $\mathbf{o}$ is a null vector, we shall say that the pencil is *axial* at T, otherwise it is *non-axial* there. For a pencil that is axial at transverse plane T Equation 14 simplifies to

$$\boldsymbol{\alpha} = \mathbf{My}\,. \tag{15}$$

Thus a set of rays is an axial pencil at a particular transverse plane if the reduced inclination of every ray is a linear function of its transverse position.

Suppose the pencil is non-axial at transverse plane T. We can imagine turning the whole pencil in space so that the ray that intersected T in Z now lies along Z. Equivalently we could choose a new longitudinal axis that lies along that ray. The effect is to subtract $\boldsymbol{\alpha}^0$ from the reduced inclination of every ray. This process converts a pencil that is non-axial into one that is axial and is always possible. It follows that, given a pencil at a transverse plane T, it is always possible to choose a longitudinal axis so that Equation 15 holds. We shall call such a longitudinal axis an axis of the pencil.

A general definition of a pencil of rays in an optical system

There are two important situations in which the conditions in Definition 1 are not satisfied: at a refracting surface and at a point or line focus. Of course at a refracting surface the medium is not homogeneous; and, as we shall see, a focus represents a singularity at which $\mathbf{M}$ does not exist or, in informal terms, is an infinite matrix. Both of these situations will be examined below. We note that Definition 1 says nothing about a pencil at a refracting surface or a focus; more particularly it does not preclude the existence of a pencil at those points. In order to be able to cope with refracting surfaces and foci, and to integrate across a whole system, we make the following definition:

DEFINITION 2 Consider a set $\mathcal{P}$ of rays in an optical system. If, by Definition 1, $\mathcal{P}$ is a *pencil* at every transverse plane, with at most a finite number of exceptions, then $\mathcal{P}$ is called a pencil at every transverse plane in the system or, simply, a pencil in the system.

It follows that a set of rays may be a pencil anywhere in an optical system, including at a thin system and a focus, even though divergency exists at neither.

So far all we have are definitions that hold under certain circumstances. We have no assurance that the definitions are either meaningful or of any use. It remains the task of the rest of this paper to attempt to show that the definitions do indeed make sense and may be useful.

A pencil of parallel rays in a homogeneous medium

The first thing we shall do is to examine the case of a pencil with $\mathbf{M} = \mathbf{O}$ D, that is, the divergency is null, at a particular transverse plane T in an optical system that consists of nothing but a homogeneous medium. Equation 14 shows that, in this case, $\boldsymbol{\alpha}^0 = \boldsymbol{\alpha}$ for every ray in the pencil. Thus all the rays in the pencil have the same reduced inclination. In other words the rays in the pencil are parallel. They neither diverge nor converge. Thus a null divergency implies a pencil of parallel rays in T. However, because of Axiom 1 exactly the same holds in every transverse plane up- and downstream from T. Hence, by Definition 1, the rays constitute a pencil at every transverse plane in the system. By Definition 2 we say that the rays constitute a pencil in the system.

Clearly the definitions are in keeping with what we would have expected. So far, therefore, the definitions make sense. We have made a good start. The next two steps, however, are a little more difficult.

A pencil across a thin system

Consider a thin system with entrance plane T_0 immediately before it and exit plane T_1 immediately after it (Figure 3). The system lies between T_0 and T_1 in the figure and is not shown explicitly. Consider a set of rays traversing the system. In T_0 a ray in the set has transverse position $\mathbf{y}_0$ and reduced inclination $\boldsymbol{\alpha}_0$ with respect to longitudinal axis Z. Suppose that the rays obey Equation 14, that is,

$$\boldsymbol{\alpha}_0 = \mathbf{M}_0\mathbf{y}_0 + \boldsymbol{\alpha}_0^0 \qquad (16)$$

where $\mathbf{M}_0$ and $\boldsymbol{\alpha}_0^0$ are particular matrices. By Definition 1 the rays constitute a pencil at T_0 and have divergency $\mathbf{M}_0$ there. We note that $\mathbf{M}_0$ may be symmetric or asymmetric.

We now invoke Axiom 2. We model the thin system with small prisms as before[8]. Refraction at the surfaces of the prisms causes deflection of the rays. Consider the small prism whose centre is located with transverse position $\mathbf{y}_0$. We write

$$\mathbf{p} = \boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0 . \qquad (17)$$

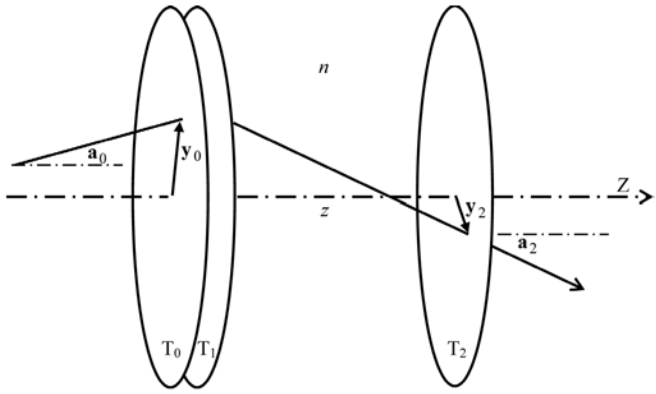

Figure 3 A pencil of rays, only one of which is shown, traversing an optical system consisting of a thin system followed by a homogeneous gap of width *z*. The thin system lies between transverse planes T_0 and T_1 and the gap between T_1 and T_2. T_0 is immediately before and T_1 immediately after the thin system.

We may call $\mathbf{p}$ the *reduced deflection* of the ray through the prism. (It is the actual deflection only when the media up- and downstream from the prism both have index 1.) We arrange the small prisms so that

$$\mathbf{p} = \mathbf{C}\mathbf{y}_0 + \boldsymbol{\pi} \qquad (18)$$

where $\mathbf{C}$ is any fixed 2×2 matrix, symmetric or asymmetric, and $\boldsymbol{\pi}$ any fixed 2×1 matrix. We call $\mathbf{C}$ the *divergence* of the thin system. This is in contrast to the *divergency* $\mathbf{M}$ of light. We call $\boldsymbol{\pi}$ the *deflectance* of the system. Equation 18 can be regarded as a generalization of Campbell's[23] Equation 1 in the sense that, for thin systems, $\mathbf{M}$ is not necessarily symmetric and $\boldsymbol{\pi}$ is not necessarily $\mathbf{o}$. When $\boldsymbol{\pi} = \mathbf{o}$ Equation 18 reduces to

$$\mathbf{p} = \mathbf{C}\mathbf{y}_0 \qquad (19)$$

and the thin system is said to be *centred* with respect to Z. When $\boldsymbol{\pi} \neq \mathbf{o}$ the system is *decentred* with respect to Z.

The system has *dioptric power* $\mathbf{F}$ defined by

$$\mathbf{F} := -\mathbf{C}\ . \tag{20}$$

The divergence **C** and the dioptric power **F** are generalizations of the corresponding concepts defined before[21, 24]. What were necessarily symmetric matrices for thin systems can now be symmetric or asymmetric.

From Equations 17 and 18 one obtains

$$\boldsymbol{\alpha}_1 = \mathbf{C}\mathbf{y}_0 + \boldsymbol{\alpha}_0 + \boldsymbol{\pi}\ . \tag{21}$$

Now $\boldsymbol{\alpha}_1$ is the reduced inclination of the ray in transverse plane T_1 immediately after the thin system. Substituting from Equation 16 into Equation 21 we obtain

$$\boldsymbol{\alpha}_1 = \mathbf{C}\mathbf{y}_0 + \mathbf{M}_0\mathbf{y}_0 + \boldsymbol{\alpha}_0^0 + \boldsymbol{\pi}\ . \tag{22}$$

Because the system is thin $\mathbf{y}_0 = \mathbf{y}_1$. Hence Equation 22 becomes

$$\boldsymbol{\alpha}_1 = (\mathbf{C} + \mathbf{M}_0)\mathbf{y}_1 + \boldsymbol{\alpha}_0^0 + \boldsymbol{\pi}\ . \tag{23}$$

Setting

$$\mathbf{C} + \mathbf{M}_0 = \mathbf{M}_1 \tag{24}$$

and

$$\boldsymbol{\alpha}_0^0 + \boldsymbol{\pi} = \boldsymbol{\alpha}_1^0 \tag{25}$$

we are able to rewrite Equation 23 as

$$\alpha_1 = \mathbf{M}_1\mathbf{y}_1 + \alpha_1^0 \tag{26}$$

which has the same form as Equation 14.

Equation 26 holds for every ray in transverse plane T_1. It follows from Definition 1 that the rays constitute a pencil at T_1 and that the pencil has divergency $\mathbf{M}_1$ there given by Equation 24. It also follows that the ray intersecting T_1 in longitudinal axis Z has reduced inclination $\boldsymbol{\alpha}_1^0$ given by Equation 25. Thus a pencil immediately upstream of a thin system implies a pencil immediately downstream of the system. Also, by Equation 25, a pencil that is axial at incidence onto a thin system is axial at emergence if and only if the lens is centred ($\boldsymbol{\pi} = \mathbf{o}$).

A pencil in a homogeneous gap

Consider now a homogeneous gap. Suppose that rays cross the gap from transverse plane T_1 to transverse plane T_2 (Figure 3) at a reduced distance ζ downstream. The objective is to show that, if they define a pencil at T_1, they also define a pencil at T_2.

We are interested in the properties $\boldsymbol{\alpha}_2$ and $\mathbf{y}_2$ of the rays in T_2 and the relationship between them. Across the gap the inclination of every ray remains unchanged (Axiom 1). Thus $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_1$. The transverse position in T_2 is given by

$$\mathbf{y}_2 = \mathbf{y}_1 + \zeta\boldsymbol{\alpha}_2 . \tag{27}$$

With Equation 26 in mind we multiply Equation 27 from the left by $\mathbf{M}_1$:

$$\mathbf{M}_1\mathbf{y}_2 = \mathbf{M}_1\mathbf{y}_1 + \zeta\mathbf{M}_1\boldsymbol{\alpha}_2 . \tag{28}$$

Subtracting Equation 26 from Equation 28 we obtain

$$\mathbf{M}_1\mathbf{y}_2 - \boldsymbol{\alpha}_1 = \zeta\mathbf{M}_1\boldsymbol{\alpha}_2 - \boldsymbol{\alpha}_1^0 . \tag{29}$$

Rearranging and making use of the fact that $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_1$ we find that

$$\boldsymbol{\alpha}_2 + \zeta\mathbf{M}_1\boldsymbol{\alpha}_2 = \mathbf{M}_1\mathbf{y}_2 + \boldsymbol{\alpha}_1^0 . \tag{30}$$

This leads to

$$(\mathbf{I} + \zeta\mathbf{M}_1)\boldsymbol{\alpha}_2 = \mathbf{M}_1\mathbf{y}_2 + \boldsymbol{\alpha}_1^0 \tag{31}$$

and

$$\boldsymbol{\alpha}_2 = (\mathbf{I} + \zeta\mathbf{M}_1)^{-1}\mathbf{M}_1\mathbf{y}_2 + (\mathbf{I} + \zeta\mathbf{M}_1)^{-1}\boldsymbol{\alpha}_1^0 \tag{32}$$

provided the inverse exists. Cases in which the inverse does not exist are interesting in their own right. They turn out to imply that the pencil shrinks to a focal point or line in transverse plane T_2. We shall consider such cases separately below; for the moment we exclude them from consideration. Under this exclusion we can set

$$(\mathbf{I} + \zeta\mathbf{M}_1)^{-1}\mathbf{M}_1 = \mathbf{M}_2 \tag{33}$$

and

$$\left(\mathbf{I}+\zeta\mathbf{M}_1\right)^{-1}\boldsymbol{\alpha}_1^0=\boldsymbol{\alpha}_2^0 \tag{34}$$

and rewrite Equation 32 as

$$\boldsymbol{\alpha}_2=\mathbf{M}_2\mathbf{y}_2+\boldsymbol{\alpha}_2^0. \tag{35}$$

Equation 35 is of the same form as Equation 14. It follows from Definition 1 that a pencil at one transverse plane implies a pencil at every transverse plane in a homogeneous gap. In particular (from Equation 34) an axial pencil is axial everywhere if it is axial anywhere. Equation 33 shows how the divergency changes across the gap.

One can express Equation 33 in three alternative forms. Because of result 3.5.2(6)(a) of Lütkepohl's handbook[25] Equation 33 can also be written

$$\mathbf{M}_1\left(\mathbf{I}+\zeta\mathbf{M}_1\right)^{-1}=\mathbf{M}_2. \tag{36}$$

(Compare this with Campbell's[14] Equation 8.) Writing out the entries of the matrices and performing the operations, one finds that Equation 36 is equivalent to

$$\frac{\mathbf{M}_1+\zeta\mathbf{I}\det\mathbf{M}_1}{\zeta^2\det\mathbf{M}_1+\zeta\operatorname{tr}\mathbf{M}_1+1}=\mathbf{M}_2 \tag{37}$$

where det and tr represent the determinant and trace of the matrix. If $\mathbf{M}_1$ is nonsingular one can also write

$$\left(\mathbf{M}_1^{-1}+\zeta\mathbf{I}\right)^{-1}=\mathbf{M}_2. \tag{38}$$

Depending on the circumstances all four of the expressions for $\mathbf{M}_2$ (Equations 33 and 36 to 38) can be useful.

Pencils in compound optical systems

It follows from what has been said above that a pencil traversing a thin system remains a pencil and that the divergency of the pencil changes according to Equation 24. It also follows that, with the exception of transverse planes where the inverse in Equation 33 does not exist, that is, at foci, a pencil traversing a homogeneous gap remains a pencil and its divergency changes according to Equation 33 or Equations 36 to 38.

Consider an optical system consisting of a finite number of successive thin systems and homogeneous gaps. It follows that a pencil of rays at one transverse plane implies that the rays define a pencil at every transverse plane except only at the component thin systems and at any foci. It is shown below that there are at most two foci associated with any homogeneous gap. Thus the exceptional transverse planes are finite in number. It follows, then, by Definition 2, that a pencil anywhere in the optical system is a pencil everywhere in the system.

Step-along calculations involving divergency can be performed across optical systems using the equations obtained above provided the exceptional transverse planes are avoided.

Generalized vergence in systems with thin elements of asymmetric power

Guided by Equation 12 we now define a generalized vergence $\mathbf{L}$ by

$$\mathbf{L}:=-\mathbf{M}. \tag{39}$$

This generalizes Fick's[11, 12] and Keating's[17, 20] vergence, which is always symmetric, to a vergence which may be symmetric or asymmetric. Incidentally Fick's interpretation of vergence, like that here for divergency, was apparently in terms of rays rather than wavefronts.

If we replace $\mathbf{M}$ in Equations 24 and 36 by $-\mathbf{L}$ we obtain generalized versions of Equations 5 and 6 respectively; they apply not only for symmetric vergence but for any vergence, including asymmetric vergence in particular. Thus Equations 5 and 6, developed in conventional linear optics, actually have a wider application; they apply to vergence in general. This means that we can use them in step-along vergence procedures through compound optical systems in general, including those with thin elements of asymmetric power.

Applying Equation 39 to Equations 33, 37 and 38 we obtain alternative expressions that describe the change in vergence across a homogeneous gap:

$$\left(\mathbf{I}-\zeta\mathbf{L}_1\right)^{-1}\mathbf{L}_1=\mathbf{L}_2,$$

$$\frac{\mathbf{L}_1 - \zeta \mathbf{I} \det \mathbf{L}_1}{\zeta^2 \det \mathbf{L}_1 - \zeta \operatorname{tr} \mathbf{L}_1 + 1} = \mathbf{L}_2 \tag{41}$$

and

$$\left(\mathbf{L}_1^{-1} - \zeta \mathbf{I}\right)^{-1} = \mathbf{L}_2 . \tag{42}$$

These expressions for vergence hold under conditions that are equivalent to the conditions under which the corresponding expressions for divergency hold. All of them are conditional upon the existence of the inverse of $\mathbf{I} - \zeta \mathbf{L}_1$ and Equation 42 is also conditional upon the existence of the inverse of $\mathbf{L}_1$.

These conditions have an important bearing on point and line foci and the transverse planes in which they lie. Before we turn to those matters we examine the components and eigenstructure of divergency.

Components of divergency

The divergency $\mathbf{M}$ of a pencil at a transverse plane is a real 2×2 matrix. We can represent it explicitly as

$$\mathbf{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} . \tag{43}$$

It has the same mathematical form as the dioptric power matrix $\mathbf{F}$. Thus every mathematical property of $\mathbf{F}$ has a counterpart in $\mathbf{M}$. We shall simply rephrase properties of dioptric power in terms of divergency.

As in the case of dioptric power[8, 14, 24, 26] the divergency can be expanded in terms of the orthonormal basis **{ I, J, K, L }**as follows:

$$\mathbf{M} = M_{\mathrm{I}}\mathbf{I} + M_{\mathrm{J}}\mathbf{J} + M_{\mathrm{K}}\mathbf{K} + M_{\mathrm{L}}\mathbf{L} \tag{44}$$

where $\mathbf{I}$ is the 2×2 identity matrix,

$$\mathbf{J} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{45}$$

$$\mathbf{K} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{46}$$

and, from here on,

$$\mathbf{L} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{47}$$

$M_{\mathrm{L}}\mathbf{L}$ is the *antisymmetric* component of the divergency $\mathbf{M}$. The rest, namely $M_{\mathrm{I}}\mathbf{I} + M_{\mathrm{J}}\mathbf{J} + M_{\mathrm{K}}\mathbf{K}$, is the symmetric component. If the antisymmetric coefficient M_{L} is not zero then the divergency is *asymmetric*. $M_{\mathrm{I}}\mathbf{I}$ is the *scalar* component of the divergency. We shall refer to $M_{\mathrm{J}}\mathbf{J}$ and $M_{\mathrm{K}}\mathbf{K}$ as the *ortho*-component and the *oblique* component respectively. The coefficients of divergency are semi-sums or semi-differences:

$$M_{\mathrm{I}} = \left(m_{11} + m_{22}\right)/2 , \tag{48}$$

$$M_{\mathrm{J}} = \left(m_{11} - m_{22}\right)/2 , \tag{49}$$

$$M_{\mathrm{K}} = \left(m_{12} + m_{21}\right)/2 , \tag{50}$$

$$M_{\mathrm{L}} = \left(m_{12} - m_{21}\right)/2 . \tag{51}$$

Compare these equations with Equations A11 to A14 published elsewhere[27].

Principal meridional divergency

Exactly as for dioptric power[28] we may, for a divergency $\mathbf{M}$, define a principal divergency M and corresponding principal meridian at angle A. Again as for power[28] we abbreviate this as 'M along A' and write it as $M\{A\}$. A principal divergency M is an eigenvalue of the divergency $\mathbf{M}$ and A is the orientation of the corresponding eigenspace of $\mathbf{M}$. For any particular divergency there are at most two distinct eigenvalues or principal divergencies; we can represent them by M_+ and M_-. It is sometimes useful to express the divergency in principal meridional form; we write the principal meridional form of divergency, or, more simply, the principal meridional divergency, as $M_+\{A_+\}M_-\{A_-\}$and read it as 'M_+ along A_+ and M_- along A_- '.

The principal meridional divergency can be obtained numerically from the divergency $\mathbf{M}$ using software that gives the eigenstructure. Alternatively one can follow the procedure presented before[28] in the case of dioptric power. We outline the procedure here. For a purely scalar divergency the problem is easy. We present it first and then consider the case of all other divergencies.

For a scalar divergency, that is, a divergency of the form

$$\mathbf{M} = M_{\mathrm{I}}\mathbf{I} \;, \tag{52}$$

there is a unique principal divergency given simply by

$$M = M_{\mathrm{I}} . \tag{53}$$

All meridians are principal meridians.

For all nonscalar divergencies the principal divergencies are given by

$$M = \tfrac{1}{2}(m_1 + m_2) \pm \tfrac{1}{2}\sqrt{\mathrm{dis}\mathbf{M}} \tag{54}$$

where we call

$$\mathrm{dis}\mathbf{M} = (m_{11} - m_{22})^2 + 4m_{21}m_{12} \tag{55}$$

the *discriminant* of the divergency **M**. The principal meridian corresponding to principal divergency M is at angle

$$A = \tan^{-1}\frac{M - m_{11}}{m_{12}} \tag{56}$$

if $m_{12} \neq 0$ D,

$$A = \tan^{-1}\frac{m_{21}}{M - m_{22}} \tag{57}$$

if $M - m_{22} \neq 0$ D and

$$A = \frac{\pi}{2} \tag{58}$$

(it is a vertical principal meridian) if both $m_{12} = 0$D and $M - m_{22} = 0$D.

Equations 48 to 51 allow one to write Equations 54 and 55 as

$$M = M_{\mathrm{I}} \pm \tfrac{1}{2}\sqrt{\mathrm{dis}\mathbf{M}} \tag{59}$$

and

$$\mathrm{dis}\mathbf{M} = 4\left(M_{\mathrm{J}}^2 + M_{\mathrm{K}}^2 - M_{\mathrm{L}}^2\right) \tag{60}$$

and the expressions (Equations 56 and 57) for the angles of the principal meridians as

$$A = \tan^{-1}\frac{M - M_{\mathrm{I}} - M_{\mathrm{J}}}{M_{\mathrm{K}} + M_{\mathrm{L}}} \tag{61}$$

if $M_{\mathrm{K}} + M_{\mathrm{L}} \neq 0$,

$$A = \tan^{-1}\frac{M_{\mathrm{K}} - M_{\mathrm{L}}}{M - M_{\mathrm{I}} + M_{\mathrm{J}}} \tag{62}$$

if $M \neq M_{\mathrm{I}} - M_{\mathrm{J}}$. Equation 58 applies when neither Equation 61 nor Equation 62 applies.

If $\mathrm{dis}\mathbf{M} > 0$ D^2 then there are two distinct principal divergencies and two corresponding principal meridians. If $\mathrm{dis}\mathbf{M} < 0$ D^2 then there are no (real) principal divergencies and no (real) principal meridians. If $\mathrm{dis}\mathbf{M} = 0$ D^2 then there is a unique principal divergency given by Equation 53 and a unique corresponding principal meridian at angle

$$A = \tan^{-1}\frac{-M_{\mathrm{J}}}{M_{\mathrm{K}} + M_{\mathrm{L}}} \tag{63}$$

if $M_{\mathrm{K}} + M_{\mathrm{L}} \neq 0$D,

$$A = \tan^{-1}\frac{M_{\mathrm{K}} - M_{\mathrm{L}}}{M_{\mathrm{J}}} \tag{64}$$

if $M_{\mathrm{J}} \neq 0$D and vertical if both $M_{\mathrm{K}} + M_{\mathrm{L}} = 0$D and $M_{\mathrm{J}} = 0$D.

Principal meridional rays

Let us call a plane containing a principal meridian and the axis of the pencil a *principal meridional plane*. We shall call a ray in a principal meridional plane a *principal meridional ray*. All other rays are *skew* rays.

Consider the rays in an axial pencil that intersect the transverse plane in a principal meridian. Because the principal divergency and the corresponding principal meridian represent the eigenstructure of **M** one has

$$\mathbf{M}\mathbf{y} = M\mathbf{y}. \tag{65}$$

This equation, together with Equation 15, then shows that, in a principal meridian,

$$\boldsymbol{\alpha} = M\mathbf{y}\,. \tag{66}$$

In other words, the reduced inclination $\boldsymbol{\alpha}$ and transverse position $\mathbf{y}$ of a ray in a principal meridian are parallel and the ray, therefore, lies in the principal meridional plane. Thus all rays in the pencil that intersect the principal meridian are in fact principal meridional rays. For a principal meridional ray one can write the scalar equivalent of Equation 66:

$$\alpha = My\,. \tag{67}$$

Principal meridional divergency across a homogeneous gap

Because the rays which intersect a transverse plane in a principal meridian are principal meridional rays it follows that the angle *A* of a principal meridian is constant across a homogeneous gap. We examine what happens to the principal divergency across the gap.

Suppose a pencil at transverse plane T_1 has a principal divergency M_1. Suppose a ray in the corresponding principal plane has transverse position $y_1 \neq 0$ m. In transverse plane T_2 at reduced distance ζ the principal divergency is M_2 and the ray's transverse position is y_2. The transverse positions are related by a scalar version of Equation 27:

$$y_2 = y_1 + \zeta\alpha \tag{68}$$

where α is the ray's constant reduced inclination. Because of Equation 67

$$M_1 y_1 = M_2 y_2\,. \tag{69}$$

Substituting from Equation 68 into Equation 69 and making use of Equation 67 we obtain

$$M_1 y_1 = M_2\left(y_1 + \zeta M_1 y_1\right) \tag{70}$$

from which we find that

$$\frac{M_1}{1+\zeta M_1} = M_2, \tag{71}$$

a scalar form of Equations 33 and 36, as might have been expected.

Locations of the focal planes

We now return to the singular cases excluded above at Equation 33. They were those cases in which $\mathbf{I} + \zeta\mathbf{M}_1$ was singular and they correspond to a zero denominator in Equation 71. Informally the principal divergency (M_2 in Equation 71) becomes infinite at the singularity. As we shall see the rays form a point or line focus. The transverse plane in which they form is a *focal* plane.

Consider a pencil in a homogeneous medium. It has divergency $\mathbf{M}$ in transverse plane T. Suppose it has a real principal divergency *M*. Equating the denominator on the left of Equation 71 to zero one finds that there is a corresponding focal plane located at reduced distance

$$\zeta = -\frac{1}{M}, \tag{72}$$

again as might have been expected. For each real and distinct principal divergency there is a real and distinct focal plane located at reduced distance given by Equation 72. Formally Equation 72 does not hold if the principal divergency is zero (the rays in the principal meridional plane are parallel); informally we can say that the focal plane is at infinity.

We turn attention now to the nature of the foci in the focal planes.

Focal points and lines

Consider an axial pencil with divergency $\mathbf{M}_1$ at transverse plane T_1. Let T_2 be a transverse plane at reduced distance ζ. Because $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2$ and because of Equation 15 Equation 27 can be written

$$\mathbf{y}_2 = \left(\mathbf{I} + \zeta\mathbf{M}_1\right)\mathbf{y}_1\,. \tag{73}$$

This gives the location $\mathbf{y}_2$ of a ray in the focal plane in terms of the ray's location $\mathbf{y}_1$ in T_1.

Equation 73 represents a mapping from ray positions in transverse plane T_1 to ray positions in transverse plane T_2. The positions that are possible in T_2 depend on the rank of the multiplier $\mathbf{I} + \zeta\mathbf{M}_1$. Being a 2×2 matrix it has rank 0, 1 or 2. Because

$\mathrm{rank}(\mathbf{I}+\zeta\mathbf{M}_1)$ is the dimension of the range[29, 30] it follows that Equation 73 defines a subspace of the plane that has dimension equal to the rank. Thus a rank of 0 implies that all the rays arrive at the same point (dimension 0) in T_2; in other words there is a focal point in T_2. Similarly a rank of 1 implies a line focus (dimension 1) in T_2. Finally a rank of 2 (that is, the matrix has full rank and, hence, its inverse exists) implies that the rays can arrive anywhere (dimension 2) in T_2. Thus a rank of 2 satisfies the proviso assumed at Equation 33. There remain only the two cases: a rank of 0 (a point focus) and a rank of 1 (a line focus). We consider each in turn.

Point foci

At a particular transverse plane a pencil has divergency **M**. A point focus is formed at a transverse plane located at reduced distance ζ if

$$\mathrm{rank}(\mathbf{I}+\zeta\mathbf{M})=0\cdot \tag{74}$$

But this implies

$$\mathbf{I}+\zeta\mathbf{M}=\mathbf{O} \tag{75}$$

which is possible only if **M** is a scalar matrix. It follows that a focal point is possible only if **M** is given by Equation 52. Furthermore the focal point is unique and is located at a reduced distance given by Equation 72.

Line focus

On the other hand a line focus is located in the transverse plane at reduced distance ζ if

$$\mathrm{rank}(\mathbf{I}+\zeta\mathbf{M})=1\cdot \tag{76}$$

This implies that

$$\mathbf{I}+\zeta\mathbf{M}=\begin{pmatrix}1+\zeta m_{11} & \zeta m_{12}\\ \zeta m_{21} & 1+\zeta m_{22}\end{pmatrix} \tag{77}$$

is singular. It follows that the rows of $\mathbf{I}+\zeta\mathbf{M}$ are linearly dependent and that the rays in the focal plane are confined to a straight line, the line focus, through axis Z. Furthermore the line focus is at angle

$$\phi=\tan^{-1}\left(\frac{\zeta m_{21}}{1+\zeta m_{11}}\right) \tag{78}$$

if $1+\zeta m_{11}\neq 0$,

$$\phi=\tan^{-1}\left(\frac{1+\zeta m_{22}}{\zeta m_{12}}\right) \tag{79}$$

if $\zeta m_{12}\neq 0$ and

$$\phi=\frac{\pi}{2} \tag{80}$$

if both $1+\zeta m_{11}=0$ and $\zeta m_{12}=0$.

Using Equations 72 and 48 to 51 one obtains

$$\phi=\tan^{-1}\left(\frac{M_K-M_L}{M_I+M_J-M}\right) \tag{81}$$

if $M\neq M_I+M_J$,

$$\phi=\tan^{-1}\left(\frac{M_I-M_J-M}{M_K+M_L}\right) \tag{82}$$

if $M_K+M_L\neq 0$ D and Equation 80 if both $M=M_I+M_J$ and $M_K+M_L=0$ D.

If there is a unique focal plane Equations 81 and 82 become

$$\phi=\tan^{-1}\left(\frac{M_K-M_L}{M_J}\right) \tag{83}$$

if $M_J\neq 0$ D,

$$\phi=\tan^{-1}\left(\frac{-M_J}{M_K+M_L}\right) \tag{84}$$

if $M_K+M_L\neq 0$ D and, again, Equation 80 if both $M_J=0$ D and $M_K+M_L=0$ D.

Suppose **M** has two real principal powers M_+ and M_- with corresponding principal meridians at angles A_+ and A_- given by Equation 61, 62 or 58. The corresponding focal lines are at angles ϕ_+ and ϕ_- given by Equation 81, 82 or 80. The trace of a matrix is the sum of its eigenvalues (see result 5.2.1(18)(b) of Lütkepohl's handbook[25]). Hence from Equation 48

$$M_+=2M_I-M_-\cdot \tag{85}$$

Substituting into Equations 81 and 82 for ϕ_+ we find that $\phi_+ = A_-$. In general we obtain

$$\phi_\pm = A_\mp . \qquad (86)$$

One can interpret this as saying that a focus in one principal meridian results in a line in the other principal meridian.

Concluding remarks

Here we have defined a pencil of rays at a transverse plane in a homogeneous medium (Definition 1), not in terms of a focus as is done in Gaussian optics, but in terms of the dependence of the reduced inclination of rays on their transverse position. This differs from Fick's interpretation[11, 12] of vergence in terms of rays only in that vergence is not constrained to being symmetric. The rays constitute a pencil if the dependence is affine. Formally a pencil of rays at a transverse plane is an equivalence class[31] determined by the divergency **M** as in Equation 14. The divergency is not defined at a refracting surface or at a focal point or focal line. Informally divergency is an infinite matrix at a focus. (Infinite symmetric vergences have been discussed elsewhere[32].)

It is important to note that, by Definition 1, a pencil at a particular transverse plane is merely a particular type of array of straight line segments, and its vergence and divergency are properties of that pencil. There is no implication of how that array might have arisen. It might have arisen by means of an array of small prisms arranged as described before[8] in a transverse plane, but that is not a requirement, and it does not exclude other possibilities. Definition 2 allows one to define a pencil in which rays are kinked according to Equation 18 at a finite number of transverse planes. It is important to note, too, that wavefronts are not involved. Of course if the pencil's vergence is symmetric then one would be able to define a smooth wavefront in the usual way. In a sense what is happening here is that we are exploring an optics freed from the constraint of a smooth wavefront. In this sense we are generalizing linear optics in particular as it is usually formulated. It becomes possible to examine sets of rays (nodal rays[33] for example) for which it may be that wavefronts have no meaning.

The negative of the divergency (Equation 39) is a generalization of Fick's[11, 12] and Keating's vergence[17, 20], which is necessarily symmetric, to a vergence which is general, that is, symmetric or asymmetric.

The behaviour of divergency across a thin system is characterized by Equation 24. The behaviour across a homogeneous gap can be expressed in several forms, Equations 33 and 36 to 38. Equation 37 is, in effect, a generalization of an equation derived by Acosta and Blendowske[34]. A pencil is defined at every transverse plane in an optical system (Definition 2). Equations 24 and 33 or 36 to 38 form the basis of step-along approaches that can be applied through optical systems that may contain thin elements of asymmetric power. They are in terms of divergency and divergence. Alternatively the equations and step-along approaches can be expressed in terms of vergence and dioptric power: Equation 5 for thin systems and Equations 6 and 40 to 42 across a homogeneous gap.

Because divergency has the same mathematical form as dioptric power the same mathematical and statistical approaches apply. Thus divergency can be split into four orthogonal components as in Equation 44 with four coefficients given by Equations 48 to 51. The eigenstructure of the divergency gives the principal divergencies, of which there are at most two, and corresponding principal meridians of the divergency. If the divergency is purely scalar (Equation 52) then every meridian is a principal meridian. If the divergency is not purely scalar then there may be two, one or no distinct (real) principal divergencies according as the discriminant of the divergency (Equations 55 or 60) is positive, zero or negative. The principal meridian is at an angle given by Equations 56, 57, 58, 61 or 62.

In a homogeneous medium a focal plane corresponds to each principal divergency its location being given by Equation 72. The focal plane contains a focal point if the divergency is scalar. Otherwise the focal plane contains a focal line at angle given by Equations 78 to 82.

Focus in one principal meridian results in a focal line lying in the other principal meridian. The principal meridians do not change across a homogeneous gap and are not generally orthogonal unless the divergency is symmetric. The line foci are also not orthogonal in general.

An accompanying paper[9] presents illustrations of pencils of rays with divergencies that are symmetric and asymmetric and shows associated point and line foci.

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References

1. Tunnacliffe AH, Hirst JG. *Optics*. Association of Dispensing Opticians, London, 1981 6.
2. Tunnacliffe AH. *Introduction to Visual Optics*. Association of Dispensing Opticians, London, 1984 8.
3. Diepes H, Blendowske R. *Optik und Technik der Brille*. Optische Fachveröffentlichung GmbH, Heidelberg, 2002 7.
4. Keating MP. *Geometric, Physical, and Visual Optics*. Butterworths, Boston 1988 16.
5. Bennett AG, Rabbetts RB. *Clinical Visual Optics.* Butterworths, London, 1984 9.
6. Freeman MH, Hull CC. Optics, 11th ed. Butterworth-Heinemann, Edinburgh, 2003 Section 3.2.
7. Smith G, Atchison DA. *The Eye and Visual Optical Instruments.* Cambridge University Press, Cambridge, 1997 76.
8. Harris WF, van Gool RD. Thin lenses of asymmetric power. *S Afr Optom* 2009 **68** 52-60.
9. Harris WF, van Gool RD. *Foci in ray pencils of general divergency. S Afr Optom* 2009 **68** 111-129.
10. Harris WF. Step-along vergence procedures in stigmatic and astigmatic systems. *Ophthal Physiol Opt* 2000 **20** 487-493.
11. Fick HH. Fortschrittliche Rechnungsarten in der Augenoptik. Folge 20. *Der Augenoptiker* 1973 (12) 55-61.
12. Fick HH. Fortschrittliche Rechnungsarten in der Augenoptik. Folge 21. *Der Augenoptiker* 1974 (2) 52-54.
13. Harris WF. Wavefronts and their propagation in astigmatic optical systems. *Optom Vis Sci* 1996 **73** 606-612.
14. Campbell CE. The refractive group. *Optom Vis Sci* 1997 **74**, 381-387.
15. Diepes H, Blendowske R. *Optik und Technik der Brille.* Optische Fachveröffentlichung GmbH, Heidelberg, 2002, 477 et seq.
16. Blendowske R. Hans-Heinrich Fick: early contributions to the theory of astigmatic systems. *S Afr Optom* 2003 **82** 105-10.
17. Keating MP. Lens effectivity in terms of dioptric power matrices. *Am J Optom Physiol Opt* 1981 **58** 1154-1160.
18. Fick HH. Fortschrittliche Rechnungsarten in der Augenoptik. Folge 9. *Der Augenoptiker* 1972 (10) 45-49.
19. Long WF. A matrix formalism for decentration problems. *Am J Optom Physiol Opt* 1976 **53** 27-33.
20. Keating MP. A system matrix for astigmatic optical systems: I. Introduction and dioptric power relations. *Am J Optom Physiol Opt* 1981 **58** 810-819.
21. Harris WF. A unified paraxial approach to astigmatic optics. *Optom Vis Sci* 1999 **76** 480-499.
22. Meyer C. *Matrix Analysis and Applied Linear Algebra.* SIAM, Philadelphia 2000 89.
23. Campbell CE. Ray vector fields. *J Opt Soc Am* A 1994 **11** 618-622.
24. Harris WF. Dioptric power: its nature and its representation in three- and four-dimensional space. *Optom Vis Sci* 1997 **74** 349-366.
25. Lütkepohl H. *Handbook of Matrices*. Wiley, Chichester, 1996.
26. Harris WF. Quantitative analysis of transformed ray transferences of optical systems in a space of augmented Hamiltonian matrices. *S Afr Optom* 2007 **66** 62-67.
27. Harris WF. Analysis of astigmatism in anterior segment surgery. *J Cataract Refract Surgery* 2001 **27** 107-128.
28. Harris WF. Interconverting the matrix and principal meridional representations of dioptric power in general including powers with nonorthogonal and complex principal meridians. *Ophthal Physiol Opt* 2001 **21** 247-252.
29. Friedberg SH, Insel AJ, Spence LE. Linear Algebra, 2nd ed. Prentice-Hall International, New York, 1992 59.
30. Horn RA, Johnson CR. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985 13.
31. Kolman B, Busby RC, Ross SC. *Discrete Mathematical Structures*, 5th ed. Pearson Prentice Hall, Upper Saddle River, NJ, 2004 Section 4.5.
32 Harris WF. Interconverting the matrix and principal-meridional representations of dioptric power and reduced vergence. *Ophthal Physiol Opt* 2000 **20** 494-500.
33. Harris WF. Nodes and nodal points and lines in eyes and other optical systems. *Ophthal Physiol Opt* 2009 **29** in press.
34. Acosta A, Blendowske R. Paraxial propagation of astigmatic wavefronts in optical systems by an augmented stepalong method for vergences. *Optom Vis Sci* 2005 **82** 923-32.