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Optical axes of catadioptric systems including visual, Purkinje and other nonvisual systems of a heterocentric astigmatic eye

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Abstract

For a dioptric system with elements which may be heterocentric and astigmatic an optical axis has been defined to be a straight line along which a ray both enters and emerges from the system. Previous work shows that the dioptric system may or may not have an optical axis and that, if it does have one, then that optical axis may or may not be unique. Formulae were derived for the locations of any optical axes. The purpose of this paper is to extend those results to allow for reflecting surfaces in the system in addition to refracting elements. Thus the paper locates any optical axes in catadioptric systems (including dioptric systems as a special case). The reflecting surfaces may be astigmatic and decentred or tilted. The theory is illustrated by means of numerical examples. The locations of the optical axes are calculated for seven optical systems associated with a particular heterocentric astigmatic model eye. The optical systems are the visual system, the four Purkinje systems and two other nonvisual systems of the eye. The Purkinje systems each have an infinity of optical axes whereas the other nonvisual systems, and the visual system, each have a unique optical axis. (*S Afr Optom* 2010 **69**(3) 152-160)

Key words: Astigmatism, catadioptric system, optical axis, transference, optical axis locator, Purkinje system, symplecticity.

Introduction

For a general dioptric system, that is, a dioptric system with elements that may be heterocentric and astigmatic, an optical axis is, by definition, a straight line along which a ray traversing the system both enters and leaves.[1] Given the length and transference of the system, and the indices of refraction of the media immediately before and after it, one can determine whether the system has an optical axis; and if the system does have an optical axis one can determine whether that optical axis is unique or not and find the location or locations of all of them in the system[1, 2]. The purpose of this paper is to generalize the definition and the results to catadioptric systems. In other words the purpose is to define optical axes for systems that may also contain reflecting surfaces which may be astigmatic and tilted or decentred. The optical model used here is linear optics.

Catadioptric systems

Let us imagine a ray traversing a general optical system. Tracing the ray through the system we observe that it undergoes a sequence of steps involving refraction and reflection. Let μ be the number of reflections. We partition the set of all systems into those for which μ is an odd number and those for which μ is even number including 0; we call the former *odd*-catadioptric and the latter *even*-catadioptric[3]. Dioptric systems are even-catadioptric because zero is an even number.

The analysis below holds for catadioptric systems in general. The theory will be illustrated numerically by application to seven optical systems associated with the eye and described recently[3]. Three of the systems are even-catadioptric (the visual system with $\mu = 0$ and two nonvisual systems with $\mu = 2$) and four odd-catadioptric (the Purkinje systems which have $\mu = 1$). The transferences of these systems have been calculated[3]. Here we draw on these previous results to locate optical axes in those systems.

Definition of an optical axis

The definition given before[1] for an optical axis in a dioptric system we simply generalize unchanged to catadioptric systems. Thus we have the following:

DEFINITION If the incident and emergent segments of a ray traversing a catadioptric system lie on the same straight line then that line is an optical axis of the system.

Nothing is said in this definition about the ray within the system. What it is doing there is of no consequence. Within the system segments of the ray that defines an optical axis may or may not lie on the optical axis.

Figure 1 shows the entrance T_0 and exit T planes of an arbitrary even-catadioptric system. The system itself is not shown. If there are no reflections, that is, if the system is dioptric ($\mu = 0$) then the system lies between T_0 and T. If there are reflections ($\mu > 0$) the system may extend upstream from T_0 and downstream from T. A ray R_0R traverses the system in Figure 1. Only the incident R_0 and emergent R segments of the ray are shown. In the figure R_0 and R happen to lie on straight line O. By definition, then, O is an optical axis of the system.

Figure 2 shows a ray traversing an odd-catadioptric system. Again segments R_0 and R lie on the same straight line which then, by definition, is an optical axis of the system. In contrast to the case of even-catadioptric systems the segments are oppositely directed.

This definition of optical axis depends on the concept of the ray[4, 5]; it is not a definition limited to linear optics. In order to examine questions of existence and location of optical axes we now invoke linear optics and so introduce the limitations associated therewith. For a treatment of linear optics and its limitations and relation to other optical models the reader is referred elsewhere[6].

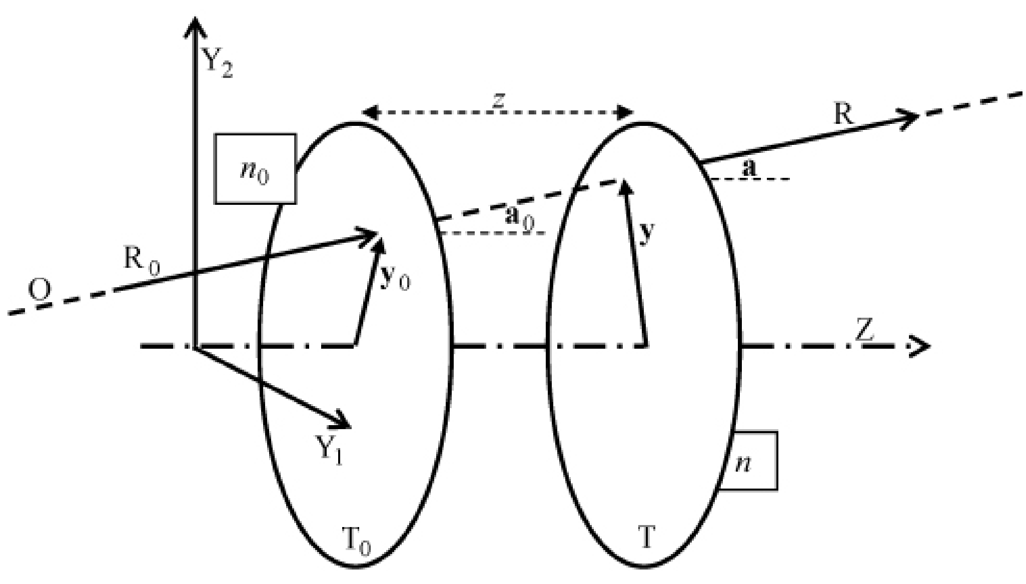

Figure 1 An optical axis O and entrance T_0 and exit T planes of an even-catadioptric system. A ray with incident segment R_0 on O has emergent segment R also on O, with the two segments pointing in the same direction. The system itself is not shown; it may extend upstream from T_0 and downstream from T.

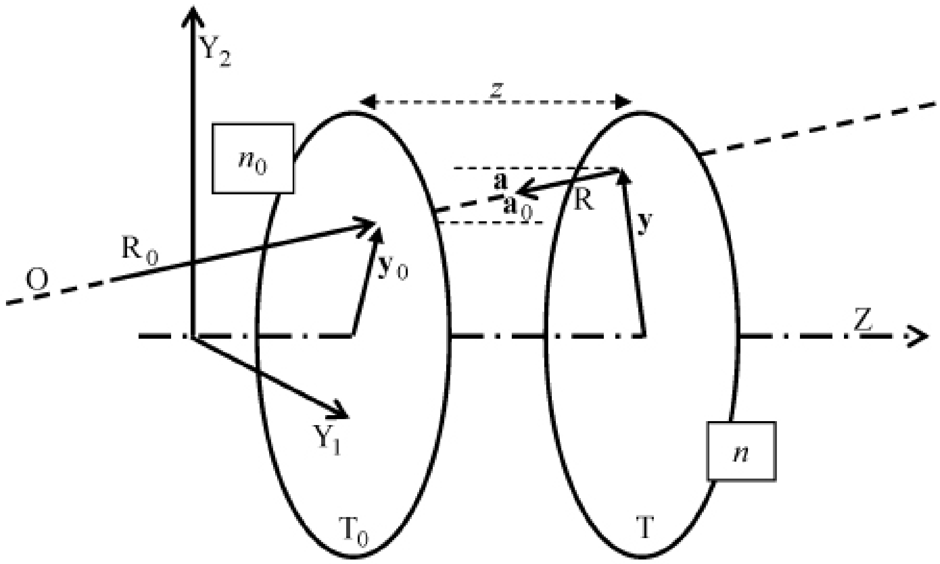

Figure 2 An optical axis O and entrance T_0 and exit T planes of an odd-catadioptric system. The incident and emergent segments of a ray lie on O but point in opposite directions. Notice that emergent segment R is in a medium of index *n* in both this figure and Figure 1.

Linear optics

Consider a ray traversing catadioptric system S (Figures 1 and 2). In linear optics the emergent transverse position $\mathbf{y}$ and the emergent inclination $\mathbf{a}$ of the ray are represented in terms of the incident transverse position $\mathbf{y}_0$ and $\mathbf{a}_0$ incident inclination by

$$\mathbf{A}\mathbf{y}_0 + n_0\,\mathbf{B}\mathbf{a}_0 + \mathbf{e} = \mathbf{y} \tag{1}$$

$$\mathbf{C}\mathbf{y}_0 + n_0\,\mathbf{D}\mathbf{a}_0 + \boldsymbol{\pi} = n\mathbf{a}\,. \tag{2}$$

(The symbolism used here is the same as that used in other papers[1-3, 7, 8].) Transverse positions $\mathbf{y}_0$ and $\mathbf{y}$ and inclinations $\mathbf{a}_0$ and $\mathbf{a}$ are all relative to longitudinal axis Z. They are 2×1 matrices with Cartesian coordinates with respect to transverse axes Y_1 and Y_2. Y_1 and Y_2 are mutually orthogonal; we shall usually think of them as horizontal and vertical respectively with Y_1 pointing to the right and Y_2 pointing upward when one is looking in the direction defined by Z. Incident segment R_0 of the ray is in a medium (outside system S) of index of refraction n_0 and emergent segment R (also outside S) is in a medium of the index *n*.

$\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{e}$ and $\boldsymbol{\pi}$ are the six *fundamental first-order optical properties* of system S. We call them the *dilation*, the *disjugacy*, the *divergence*, the *divarication*, the *translation* and the *deflectance* of S respectively. The first four are 2×2 matrices. The last two are 2×1 matrices; they account for the effects of tilt, decentration or prism in the system. They are submatrices of the transference $\mathbf{T}$ (5×5) and arranged as follows

$$\mathbf{T} = \begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{e} \\ \mathbf{C} & \mathbf{D} & \boldsymbol{\pi} \\ \mathbf{o}^{\mathrm{T}} & \mathbf{o}^{\mathrm{T}} & 1 \end{pmatrix} \tag{3}$$

where $\mathbf{o}$ is the 2×1 null matrix and $\mathbf{o}^{\mathrm{T}}$ its matrix transpose. We define the 4×1 matrix $\boldsymbol{\delta}$ by

$$\boldsymbol{\delta} = \begin{pmatrix} \mathbf{e} \\ \boldsymbol{\pi} \end{pmatrix}. \tag{4}$$

$\mathbf{T}$ is an augmented symplectic matrix. (Important consequences of symplecticity are summarized elsewhere.[9])

Suppose all the optical elements in system S are centred on axis Z; none is tilted or decentred. A ray incident along Z will traverse and emerge from the system along Z. Thus Z is an optical axis of S. For that ray $\mathbf{y}_0$ and $\mathbf{a}_0$ are both null matrices and so are $\mathbf{y}$ and $\mathbf{a}$. It follows from Equations 1 and 2 that $\mathbf{e}$ and $\boldsymbol{\pi}$ must also be (2×1) null matrices. In this case then the system has

$$\boldsymbol{\delta} = \mathbf{o} \tag{5}$$

where $\mathbf{o}$ is a 4×1 null matrix.

When the elements of S are not centred on Z, or are tilted, then, usually, $\boldsymbol{\delta}$ is not null. A ray incident along Z emerges with transverse position $\mathbf{y} = \mathbf{e}$ and inclination $\mathbf{a} = \boldsymbol{\pi}/n$ as is clear from Equations 1 and

2 respectively. In other words matrix $\boldsymbol{\delta}$ of system S is a measure of the transverse displacement away from longitudinal axis Z undergone by a ray incident onto S along Z. For want of a name we shall call $\boldsymbol{\delta}$ the *transversion* of the system.

Even-catadioptric systems

The distance parallel to Z from T_0 to T of the even-catadioptric system in Figure 1 is z. We shall call z the *effective length* of the system. z may or may not be the actual length of the optical instrument; because of reflection within it the instrument may extend upstream of T_0 and downstream of T. z may be positive (as in Figure 1), zero or negative.

From Figure 1 one sees that, for the ray that defines optical axis O,

$$\mathbf{a}_0 = \mathbf{a}. \tag{6}$$

Under the assumptions of linear optics it also follows that

$$\mathbf{y}_0 + z\mathbf{a}_0 = \mathbf{y}. \tag{7}$$

Substituting from these equations into Equations 1 and 2 and rearranging we obtain

$$(\mathbf{A}-\mathrm{I})\mathbf{y}_0 + (n_0\mathbf{B}-z\mathrm{I})\mathbf{a}_0 = -\mathbf{e} \tag{8}$$

and

$$\mathbf{C}\mathbf{y}_0 + (n_0\mathbf{D}-n\mathbf{I})\mathbf{a}_0 = -\boldsymbol{\pi}. \tag{9}$$

It is convenient to combine Equations 8 and 9 into the single equation

$$\begin{pmatrix} \mathbf{A}-\mathbf{I} & n_0\mathbf{B}-z\mathbf{I} \\ \mathbf{C} & n_0\mathbf{D}-n\mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{y}_0 \\ \mathbf{a}_0 \end{pmatrix} = -\begin{pmatrix} \mathbf{e} \\ \boldsymbol{\pi} \end{pmatrix}. \tag{10}$$

We now make the definitions

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}-\mathbf{I} & n_0\mathbf{B}-z\mathbf{I} \\ \mathbf{C} & n_0\mathbf{D}-n\mathbf{I} \end{pmatrix} \tag{11}$$

and

$$\mathbf{d}_0 = \begin{pmatrix} \mathbf{y}_0 \\ \mathbf{a}_0 \end{pmatrix}. \tag{12}$$

Then Equation 10 can be written

$$\mathbf{P}\mathbf{d}_0 = -\boldsymbol{\delta}. \tag{13}$$

This is the same as the result obtained before[1] for dioptric systems ($\mu = 0$). In other words the result for dioptric systems now generalizes to all even-catadioptric systems ($\mu = 0, 2, 4, \ldots$).

We shall call $\mathbf{P}$ the *locator* and $\mathbf{d}_0$ the *location* of optical axis O in system S. $\mathbf{d}_0$ is with respect to longitudinal axis Z and at entrance plane T_0.

Given a particular system S the problem of finding its optical axes reduces to finding solutions $\mathbf{d}_0$ (if any) to Equation 13. Before seeking solutions we consider odd-catadioptric systems.

Odd-catadioptric systems

Consider now the odd-catadioptric system of Figure 2. The ray leaves system S in the opposite direction to its direction at incidence. So, instead of Equation 6, we have

$$\mathbf{a}_0 = -\mathbf{a}. \tag{14}$$

Equation 7 applies as before. Substituting into Equations 1 and 2 one obtains Equation 8 as before but instead of Equation 9 one obtains

$$\mathbf{C}\mathbf{y}_0 + (n_0\mathbf{D}+n\mathbf{I})\mathbf{a}_0 = -\boldsymbol{\pi}. \tag{15}$$

The result is Equation 13 again except that locator $\mathbf{P}$ is now defined by

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}-\mathbf{I} & n_0\mathbf{B}-z\mathbf{I} \\ \mathbf{C} & n_0\mathbf{D}+n\mathbf{I} \end{pmatrix} \tag{16}$$

instead of Equation 11.

Optical axis locator

The optical axis locator $\mathbf{P}$ for odd-catadioptric systems (Equation 16) differs from the locator for even-catadioptric systems (Equation 11) only in that it has a plus sign instead of a minus sign in the bottom-right block. The two equations can be combined in one as

$$\mathbf{P} = \begin{pmatrix} \mathbf{A}-\mathbf{I} & n_0\mathbf{B}-z\mathbf{I} \\ \mathbf{C} & n_0\mathbf{D}-(-1)^{\mu}n\mathbf{I} \end{pmatrix}. \tag{17}$$

Alternatively one could use Equation 11 for both even- and odd-catadioptric systems provided that, for odd-catadioptric systems, one adopted the trick of substituting $-n$ for n. In this paper we prefer to use the more clumsy-looking but, perhaps, safer Equation 17.

Singularity of the optical axis locator

Whether or not the optical axis locator P is singular will become an important issue below. We therefore examine conditions under which it is singular.

The symplectic unit matrix[9]

$$\mathbf{E} = \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{I} & \mathbf{O} \end{pmatrix} \tag{18}$$

is nonsingular. Making use of Equations 17 and 18 one finds that

$$\mathbf{P}^{\mathrm{T}}\mathbf{E}\mathbf{P} = \begin{pmatrix} \mathbf{C}^{\mathrm{T}} - \mathbf{C} & z\mathbf{C}^{\mathrm{T}} - n_0(\mathbf{D}-\mathbf{I}) - (-1)^{\mu} n(\mathbf{A}^{\mathrm{T}} - \mathbf{I}) \\ -z\mathbf{C} + n_0(\mathbf{D}^{\mathrm{T}} - \mathbf{I}) + (-1)^{\mu} n(\mathbf{A}-\mathbf{I}) & n_0 z(\mathbf{D}^{\mathrm{T}} - \mathbf{D}) - (-1)^{\mu} n_0 n(\mathbf{B}^{\mathrm{T}} - \mathbf{B}) \end{pmatrix}. \tag{19}$$

In obtaining Equation 19 use has also been made of symplecticity[9] and, more particularly, of the facts that $\mathbf{A}^{\mathrm{T}}\mathbf{C}$ and $\mathbf{B}^{\mathrm{T}}\mathbf{C}$ are symmetric and that $\mathbf{A}^{\mathrm{T}}\mathbf{D} - \mathbf{C}^{\mathrm{T}}\mathbf{B} = \mathbf{I}$.

The optical axis locator **P** is singular or nonsingular according as $\mathbf{P}^{\mathrm{T}}\mathbf{E}\mathbf{P}$ is singular or nonsingular. (This is a consequence of Equations 5 and 6 of a previous paper[9].)

Consider an odd-catadioptric system with $n = n_0$ and $z = 0$. For such a system Equation 19 reduces to

$$\mathbf{P}^{\mathrm{T}}\mathbf{E}\mathbf{P} = \begin{pmatrix} \mathbf{C}^{\mathrm{T}} - \mathbf{C} & n_0(\mathbf{A}^{\mathrm{T}} - \mathbf{D}) \\ n_0(\mathbf{D}^{\mathrm{T}} - \mathbf{A}) & n_0^2(\mathbf{B}^{\mathrm{T}} - \mathbf{B}) \end{pmatrix}. \tag{20}$$

Suppose, in addition, that the system is such that **C** and **B** are symmetric and $\mathbf{D} = \mathbf{A}^{\mathrm{T}}$. Equation 20 reduces to

$$\mathbf{P}^{\mathrm{T}}\mathbf{E}\mathbf{P} = \mathbf{O}. \tag{21}$$

Hence, for optical systems of this class, the optical axis locator **P** is singular. Among the systems of this class are the Purkinje systems. Numerical examples are presented in the appendix to a previous paper[3] (available at http://links.lww.com/OPX/A29). (A paper in preparation provides a general proof.)

Location of the optical axis

We are now in a position to solve Equation 13 for the location $\mathbf{d}_0$ of an optical axis.

If for a particular system the optical axis locator **P** is nonsingular then we can pre-multiply both sides of Equation 13 by the inverse of **P** to obtain the unique solution

$$\mathbf{d}_0 = -\mathbf{P}^{-1}\boldsymbol{\delta}. \tag{22}$$

It follows that such a system has a single optical axis, its location being given by Equation 22.

For a system for which the locator **P** is singular the mathematics is messier. In such cases the inverse $\mathbf{P}^{-1}$ does not exist and Equation 22 does not hold. The system may or may not have an optical axis; it has an optical axis if and only if [10, 11]

$$\mathbf{P}\mathbf{P}^{-}\boldsymbol{\delta} = \boldsymbol{\delta} \tag{23}$$

where $\mathbf{P}^{-}$ is the Moore-Penrose inverse of **P**. If Equation 23 holds then all of the system's optical axes are defined by

$$\mathbf{d}_0 = -\mathbf{P}^{-}\boldsymbol{\delta} + \left(\mathbf{I} - \mathbf{P}^{-}\mathbf{P}\right)\mathbf{g}. \tag{24}$$

Here **I** is the 4×4 identity matrix and **g** is an arbitrary 4×1 matrix we may call the *generator*. Choosing a particular **g** one generates the location $\mathbf{d}_0$ of an optical axis. Choosing all possible generators **g** one locates all the optical axes of system S. (When the inverse does exist $\mathbf{P}^{-} = \mathbf{P}^{-1}$, Equation 23 is always satisfied and Equation 24 reduces to Equation 22.)

In keeping with common mathematical terminology we say that, for a particular optical system, an optical axis *exists* if Equation 23 is satisfied and *exists uniquely* if Equation 22 holds.

Routine for locating optical axes

We can now lay out a general procedure for determining whether a catadioptric system has any optical axes and, if it does, for locating all of them.

At the outset we need four items of information about the system: its transference **T**, its effective length z (that is, the distance from its entrance plane to its exit plane T_0), the indices of refraction n_0 and n and whether the number μ of reflections in it is odd or even. **T** can be calculated from a knowledge of the structure of the system as described elsewhere[3]. Software such as Matlab is ideal for handling the matrices.

One begins by calculating the locator **P** by means of Equation 17 in which **A**, **B**, **C** and **D** are obtained from **T**. The transversion $\boldsymbol{\delta}$ is also obtained from **T** (according to Equation 4).

Next one needs to determine if **P** is singular. In some cases this is obvious as in the case of the thin prism described below and in the case of Purkinje systems; in other cases one has to evaluate the determinant or rank of **P**. (If the determinant is 0 or the rank is less than 4 then **P** is singular.)

If **P** turns out to be nonsingular then the system has a unique optical axis; its location $\mathbf{d}_0$ is determined using Equation 22. (In Matlab the inverse is given by $inv(\bullet)$.)

On the other hand if **P** turns out to be singular then one has to check whether Equation 23 is satisfied. This requires the Moore-Penrose inverse $\mathbf{P}^-$ (given by $pinv(\bullet)$ in Matlab and very tedious to calculate if done by hand).

If Equation 23 is not satisfied then the system has no optical axis.

If Equation 23 is satisfied then the system has at least one optical axis. All of the optical axes can be located using Equation 24. The generator **g** contains four numbers which take on all possible real values. In particular setting all four numbers in **g** equal to 0 gives the location

$$\mathbf{d}_0 = -\mathbf{P}^-\boldsymbol{\delta} \tag{25}$$

of one of the optical axes.

Equation 12 shows that the top half of the location $\mathbf{d}_0$ is the transverse position $\mathbf{y}_0$ of the optical axis at the entrance plane T_0 and the bottom half is the inclination $\mathbf{a}_0$ of the optical axis, both $\mathbf{y}_0$ and $\mathbf{a}_0$ being relative to longitudinal axis Z.

Numerical results for seven ocular systems of a model eye

Table 1 shows the results calculated for seven ocular systems of the heterocentric astigmatic model eye considered elsewhere[3]. S_{P0} represents the visual system of the eye, an even-catadioptric system ($\mu = 0$); it is the dioptric system from entrance plane T_0 just in front of the cornea to exit plane T just in front of the retina. S_{P1} to S_{P6} are six nonvisual ocular systems. S_{P1} to S_{P4} are the four Purkinje systems, odd-catadioptric systems ($\mu = 0$) with reflection off the anterior (S_{P1}) and posterior (S_{P2}) surfaces of the cornea and the anterior (S_{P3}) and posterior (S_{P4}) surfaces of the lens of the eye; they are the systems responsible for the first to fourth Purkinje images. Their entrance and exit planes coincide and are immediately in front of the cornea. S_{P5} and S_{P6} are two nonvisual ocular systems Tshcerning[12] describes as being responsible for harmful rays reaching the retina. They are even-catadioptric systems ($\mu = 2$) with the same entrance and exit planes as the visual system; there is anterior reflection off the anterior (S_{P5}) and posterior (S_{P6}) surfaces of the lens of the eye followed by posterior reflection off the anterior surface of the cornea.

The optical axis locator **P**, listed in Table 1 for each system, is calculated using Equation 17. The top-right block of four entries are in millimetres (mm); the bottom-left entries are in reciprocal millimetres, that is, kilodioptres (kD). For all of the systems $n_0 = 1$ (light enters from air). n is the index of the medium in front of the retina (taken as 1.336 here) in the case of systems S_{P0}, S_{P5} and S_{P6}; those systems have the same effective length namely $z = 24.6$ mm. For the Purkinje systems, systems S_{P1} to S_{P4}, $n = n_0 = 1$ and $z = 0$. Properties **A**, **B**, **C** and **D** are obtained from the transferences of the systems presented before[3]. The transversions $\boldsymbol{\delta}$ are also obtained from the transferences.

For the visual system S_{P0}, and for nonvisual systems S_{P5} and S_{P6}, the locators **P** are all nonsingular (they have rank 4); thus each has a unique optical axis, its location $\mathbf{d}_0$ (calculated by means of Equation 22) being listed in the last column of Table 1. For

Table 1 The locator **P** for and location $\mathbf{d}_0$ of the optical axes of seven optical systems in a heterocentric astigmatic model eye. The entries in the top-right 2×2 block of **P** and the top two entries of $\mathbf{d}_0$ are in millimetres and entries in the bottom-left block of **P** are in kilodioptres.

System	Optical axis locator $\mathbf{P}$	Optical axis location $\mathbf{d}_0$
S_{P0}	$\begin{pmatrix} -1.0235 & -0.0174 & -7.5272 & -0.1064 \\ -0.0171 & -1.0232 & -0.1064 & -7.8163 \\ -0.0598 & -0.0013 & -0.4224 & -0.0080 \\ -0.0013 & -0.0608 & -0.0080 & -0.4451 \end{pmatrix}$	$\begin{pmatrix} 0.0486 \\ 0.3251 \\ -0.0111 \\ -0.0608 \end{pmatrix}$
S_{P1}	$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2632 & 0 & 2 & 0 \\ 0 & 0.2500 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} -0.0013 \\ -0.0025 \\ -0.0098 \\ -0.0197 \end{pmatrix} + \begin{pmatrix} 0.9830 & 0.0000 & -0.1293 & 0.0000 \\ 0.0000 & 0.9846 & -0.0000 & -0.1231 \\ -0.1293 & 0.0000 & 0.0170 & -0.0000 \\ 0.0000 & -0.1231 & -0.0000 & 0.0154 \end{pmatrix}\mathbf{g}$
S_{P2}	$\begin{pmatrix} 0.1269 & 0.0033 & 0.8667 & 0.0013 \\ 0.0033 & 0.1253 & 0.0013 & 0.8651 \\ 0.3116 & 0.0080 & 2.1269 & 0.0033 \\ 0.0080 & 0.3079 & 0.0033 & 2.1253 \end{pmatrix}$	$\begin{pmatrix} 0.0006 \\ 0.0047 \\ 0.0036 \\ 0.0323 \end{pmatrix} + \begin{pmatrix} 0.9790 & -0.0010 & -0.1434 & -0.0033 \\ -0.0010 & 0.9794 & -0.0033 & -0.1419 \\ -0.1434 & -0.0033 & 0.0210 & 0.0010 \\ -0.0033 & -0.1419 & 0.0010 & 0.0206 \end{pmatrix}\mathbf{g}$
S_{P3}	$\begin{pmatrix} 0.3864 & 0.0466 & 7.2764 & 0.1391 \\ 0.0462 & 0.4034 & 0.1391 & 7.2756 \\ 0.1267 & 0.0153 & 2.3864 & 0.0462 \\ 0.0153 & 0.1333 & 0.0466 & 2.4034 \end{pmatrix}$	$\begin{pmatrix} 0.0022 \\ -0.0004 \\ 0.0435 \\ -0.0122 \end{pmatrix} + \begin{pmatrix} 0.9972 & -0.0006 & -0.0528 & -0.0053 \\ -0.0006 & 0.9969 & -0.0053 & -0.0552 \\ -0.0528 & -0.0053 & 0.0028 & 0.0006 \\ -0.0053 & -0.0552 & 0.0006 & 0.0031 \end{pmatrix}\mathbf{g}$
S_{P4}	$\begin{pmatrix} -2.2271 & -0.1662 & -1.6771 & -1.2068 \\ -0.1631 & -2.8057 & -1.2068 & -5.8481 \\ -0.3016 & -0.0224 & -0.2271 & -0.1631 \\ -0.0224 & -0.3866 & -0.1662 & -0.8057 \end{pmatrix}$	$\begin{pmatrix} -0.1108 \\ 0.0471 \\ -0.0619 \\ 0.0542 \end{pmatrix} + \begin{pmatrix} 0.3144 & 0.1372 & -0.4432 & 0.0169 \\ 0.1372 & 0.7874 & 0.0169 & -0.3851 \\ -0.4432 & 0.0169 & 0.6856 & -0.1372 \\ 0.0169 & -0.3851 & -0.1372 & 0.2126 \end{pmatrix}\mathbf{g}$
S_{P5}	$\begin{pmatrix} -5.0992 & 0.0497 & -33.7976 & 0.0788 \\ 0.0634 & -4.6855 & 0.1264 & -31.7452 \\ -0.3006 & -0.0001 & -2.2544 & -0.0056 \\ 0.0007 & -0.2793 & -0.0030 & -2.1488 \end{pmatrix}$	$\begin{pmatrix} -0.7229 \\ -0.1931 \\ 0.0876 \\ 0.0055 \end{pmatrix}$
S_{P6}	$\begin{pmatrix} -0.6063 & 0.3935 & -37.9681 & 2.8041 \\ 0.3318 & 0.1308 & 2.4145 & -30.2289 \\ 0.0930 & 0.0313 & -1.9530 & 0.2237 \\ 0.0277 & 0.1676 & 0.2025 & -1.2863 \end{pmatrix}$	$\begin{pmatrix} -0.1246 \\ 0.2318 \\ 0.0070 \\ -0.0491 \end{pmatrix}$

example the optical axis of the visual system (S_{P0}) intersects the entrance plane just in front of the cornea at the point with horizontal and vertical coordinates 0.0486 mm and 0.3251 mm (to the right and above) longitudinal axis Z respectively. Its inclination has horizontal coordinate -0.0111 and vertical coordinate -0.0608 (radians) also relative to Z; into the eye the optical axis slopes to the left and down.

The locators **P** of the Purkinje systems are singular (they each have rank 2 in fact). Equation 23 is satisfied for each of the systems. Hence each of the systems has at least one optical axis. In fact the Purkinje systems have an infinity of optical axes as represented in the last column of Table 1 in the form of Equation 24, the 4×1 column matrix being $-\mathbf{P}^{-}\boldsymbol{\delta}$ (the location of one of the optical axes) and the 4×4 coefficient of generator **g** being $\mathbf{I}-\mathbf{P}^{-}\mathbf{P}$.

A simple example of a system for which the condition of existence of an optical axis (Equation 23) is not satisfied is a thin prism in air. For it $n_0 = n = 1$, $z = 0$, $\mu = 0$, **B** and **C** are null and **A** and **D** are identity matrices. Equation 17 reduces to $\mathbf{P} = \mathbf{O}$ from which one finds that $\mathbf{P}^{-} = \mathbf{O}$. Hence the left-hand side of Equation 23 is null and, therefore, not equal to the right-hand side unless the prism has null deflectance $\boldsymbol{\pi}$.

Concluding remarks

This paper generalizes the result obtained before[1] for dioptric systems with elements that may be heterocentric and astigmatic to catadioptric systems of which dioptric systems are a special case. The expression for the optical axis locator **P** for dioptric systems holds in fact for all even-catadioptric systems (Equation 11). For odd-catadioptric systems the locator is given by Equation 16. The two equations combine as the single equation, Equation 17.

An optical system may or may not have an optical axis, and, if it does have one that axis may or may not be unique. A system does have an optical axis if and only if Equation 23 (the condition of existence) is satisfied. The system has only one optical axis if its optical axis locator **P** is nonsingular (the condition of uniqueness). (Its location is given by Equation 22.) This is the case for the visual system and for nonvisual systems S_{P5} and S_{P6} of the eye examined here. One expects the same to be true for any eye. For Purkinje systems the locator **P** is singular. For the eye examined here there is an infinity of optical axes for each Purkinje system and one expects this also to be true of any eye.

We mention in passing that Purkinje is a common misspelling of the name of Czech scientist J. E. Purkyně (1787-1869). It is one of nine different spellings of the name that John has encountered in the literature.[13]

When we say here that an optical axis exists and has some particular location we are taking no account of stops or apertures in the system. Furthermore the results are subject to the limitations of the optical model (linear optics) that has been used. The definition of optical axis itself, however, is limited only in so far as the concept of the ray is limited.

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