S Afr Optom 2010 **69**(2) 51-57

Special rays and structures in general optical systems: generalized magnifications associated with the fundamental properties

WF Harris

Department of Optometry, University of Johannesburg, PO Box 524, Auckland Park, 2006 South Africa

<wharris@uj.ac.za>

Received 4 January 2010; revised version accepted 8 June 2010

Abstract

In a previous study special rays and structures, including nodal, principal and focal rays and structures, were defined in terms of generalized magnification of reduced inclination of rays across the system. The mathematical structure of the transference, however, suggests that there are another three possible approaches. These involve generalized magnification of transverse position of rays across the system and two cross-magnifications, one of reduced inclination to transverse position and the other of transverse position to reduced inclination. The purpose of this paper is to develop these other approaches. Nodal, principal and focal rays and structures, and, indeed, all special rays and structures, are defined in each of these approaches. Equations relating the four types of generalized magnification are derived. The approaches described here may have application in contexts where transverse position is of interest. In an appendix the generalized magnifications are calculated for nodal rays across a particular model eye. (*S Afr Optom* 2010 **69**(2) 51-57)

Key words: symplecticity, cardinal points, nodal rays, transference, astigmatism

Introduction

Gauss[1] and Listing[2] defined cardinal points for optical systems containing stigmatic and homocentric refracting elements. A recent paper[3] generalizes these concepts in two ways: it defines a large class of special points of which the cardinal points are particular cases and it defines special structures in optical systems whose refracting elements may be astigmatic and heterocentric. Underlying symmetry in the mathematics, however, suggests that the approach adopted in that paper is only one of four possible approaches to the definition of special rays and structures in general linear systems. The purpose of this paper is to explore all four approaches and obtain relationships among them.

The previous approach is one that we might describe as related to submatrix $\mathbf{D}$ in transference $\mathbf{S}=\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}$. The other three are related in a similar way to $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ instead.

A ray traversing optical system S (Figure 1) in linear optics obeys the pair of equations

$$\mathbf{A}\mathbf{y}_0+\mathbf{B}\boldsymbol{\alpha}_0=\mathbf{y}\,, \tag{1}$$

$$\mathbf{C y}_0 + \mathbf{D}\boldsymbol{\alpha}_0 = \boldsymbol{\alpha} \tag{2}$$

This paper uses the same symbolism and relationships defined in earlier papers[3, 4]. **A**, **B**, **C** and **D** are 2×2 matrices representing the dilation, the disjugacy, the divergence and the divarication of system S. They are submatrices of the system's transference **S**. $\mathbf{y}_0$ and **y** are the transverse position vectors of the ray at incidence onto and emergence from S respectively; $\boldsymbol{\alpha}_0$ and $\boldsymbol{\alpha}$ are the reduced inclinations of the ray at incidence and emergence. They are related to the (unreduced) inclinations $\mathbf{a}_0$ and **a** at incidence and emergence by

$$\boldsymbol{\alpha}_0 := n_0 \mathbf{a}_0 \tag{3}$$

$$\boldsymbol{\alpha} := n\mathbf{a} \tag{4}$$

where n_0 and n are the indices immediately up- and downstream, respectively, from S.

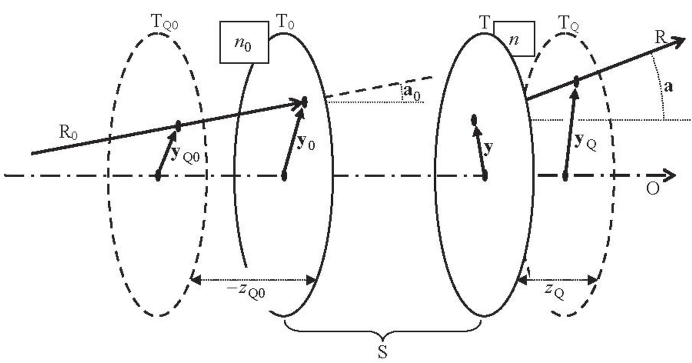

Figure 1. Ray R_0R traversing arbitrary linear optical system S. O is its optical axis and T_0 and T its entrance and exit planes respectively. Segment R_0 is incident onto S at $\mathbf{y}_0$ and with inclination $\mathbf{a}_0$; segment R emerges with transverse position y and inclination **a**. T_{Q0} and T_Q have longitudinal positions $z_{Q0} < 0$ and $z_Q > 0$ relative to T_0 and T respectively. R_0 intersects T_{Q0} in transverse position $\mathbf{y}_{Q0}$ and R intersects T_Q in $\mathbf{y}_Q$.

Equations 1 and 2 apply to all rays traversing S. In addition the previous paper[3] selected from all the rays those particular rays which obey

$$\mathbf{X}\boldsymbol{\alpha}_0 = \boldsymbol{\alpha} \tag{5}$$

for some particular matrix **X**. **X** is a generalized magnification for reduced inclination of rays across the system. The special rays for that particular **X** obey all three Equations 1, 2 and 5.

Corresponding special structures were then defined in terms of the intersections of the special rays with transverse planes before and after the system. Special structures depend on matrix **X**; for example principal structures have $\mathbf{X} = \mathbf{I}$ where **I** is an identity matrix and nodal structures have $\mathbf{X} = \mathbf{I}n / n_0$. Special structures may be points or may contain lines; nodal, principal and focal points and lines are examples. Special lines may even be what one might call nonreal lines; the lines are definable mathematically in terms of complex numbers but they have no physical existence as lines in the usual sense of the term.

The four generalized magnifications for a ray traversing a system

Inspection of Equations 1, 2 and 5 reveals an association between **X** with **D**. They have in common the fact that both operate on $\boldsymbol{\alpha}_0$ to give or contribute to $\boldsymbol{\alpha}$. Equations 1 and 2, however, have a mathematical symmetry beyond the physical meaning of the terms, which suggests that what applies to **D** should just as well apply to the other three fundamental properties, namely **A**, **B** and **C**. Accordingly we define four matrices $\mathbf{X_A}$, $\mathbf{X_B}$, $\mathbf{X_C}$ and $\mathbf{X_D}$, where the subscript specifies the fundamental property with which the matrix is associated, and we write the four equations

$$\mathbf{X_A y}_0 = \mathbf{y}, \tag{6}$$

$$\mathbf{X_B}\boldsymbol{\alpha}_0 = \mathbf{y}, \tag{7}$$

$$\mathbf{X_C y}_0 = \boldsymbol{\alpha}, \tag{8}$$

$$\mathbf{X_D}\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}. \tag{9}$$

Equation 9 merely repeats Equation 5 but it now has **X** replaced by $\mathbf{X_D}$.

The thinking we are applying here is in the spirit described in 1870 by Maxwell[5]: we are 'dealing with systems of quantities, in which the mathematical forms of the relations of the quantities are the same…, though the physical nature of the quantities may be utterly different.'

Just as $\mathbf{X_D}$ is a generalized magnification which magnifies reduced inclination of rays across the system $\mathbf{X_A}$ is a generalized magnification that magnifies transverse position of the rays across the system. $\mathbf{X_B}$ and $\mathbf{X_C}$ are cross-magnifications that 'magnify' reduced inclination to transverse position in the case of $\mathbf{X_B}$ and transverse position to reduced inclination in the case of $\mathbf{X_C}$.

Sets of special rays associated with the four magnifications

The previous paper[3] explored special rays that obeyed Equation 9. In this paper we do the same but for special rays that obey each of Equations 6, 7 and 8. Rays through a particular system which satisfy Equation 6 for a particular matrix $\mathbf{X_A}$ we shall call special rays of generalized magnification $\mathbf{X_A}$. Similarly those that satisfy Equations 7, 8 and 9 are special rays of generalized magnification $\mathbf{X_B}$, $\mathbf{X_C}$ and $\mathbf{X_D}$ respectively.

A ray along the optical axis has $\mathbf{y}_0 = \mathbf{o}$ and $\boldsymbol{\alpha}_0 = \mathbf{o}$ and is a special ray for all generalized magnifications. If the system is afocal ($\mathbf{C} = \mathbf{O}$) then that ray is the only special ray of $\mathbf{X_D} \neq \mathbf{D}$ while every ray is a special ray of $\mathbf{X_D} = \mathbf{D}$. There are no special structures in these cases. The same happens for systems with $\mathbf{A} = \mathbf{O}$, $\mathbf{B} = \mathbf{O}$ and $\mathbf{D} = \mathbf{O}$. We exclude these exceptional cases in what follows.

Consider a transverse plane T_{Q0} with longitudinal position z_{Q0} relative to entrance plane T_0 of system S (Figure 1). The incident segment of a ray has inclination $\mathbf{a}_0$ and intersects T_0 and T_{Q0} in points with transverse positions $\mathbf{y}_0$ and $\mathbf{y}_{Q0}$ respectively. Then

$$\mathbf{y}_0 = \mathbf{y}_{Q0} - z_{Q0}\mathbf{a}_0 \tag{10}$$

which repeats Equation 34 of the previous paper[3]. Similarly for the emergent segment of the ray

$$\mathbf{y} = \mathbf{y}_Q - z_Q\mathbf{a}\,; \tag{11}$$

$\mathbf{a}$ is its reduced inclination, $\mathbf{y}$ its transverse position in exit plane T of the system and $\mathbf{y}_Q$ its position in transverse plane T_Q which is located at longitudinal position z_Q relative to T. (Equation 11 is Equation 38 of the previous paper.)

We now examine the intersection of special rays of some particular generalized magnification $\mathbf{X_A}$ with transverse planes T_{Q0} and T_Q. These intersections define special structures that may be or contain singular points or lines and include focal, nodal and principal points and lines.

Special structures

First we consider the incident segments of the special rays. Substituting from Equations 3, 6 and 10 into Equation 1 and solving we obtain

$$\mathbf{y}_{Q0} = \left(\mathbf{I}z_{Q0} - \mathbf{Z}_{\mathbf{A}Q0}\right)\mathbf{a}_0 \tag{12}$$

where

$$\mathbf{Z}_{\mathbf{A}Q0} = n_0\left(\mathbf{A} - \mathbf{X_A}\right)^{-1}\mathbf{B}\,. \tag{13}$$

These two equations are the counterparts for $\mathbf{A}$ of the corresponding equations (Equations 37 and 36, respectively, of the previous paper) for $\mathbf{D}$. $\mathbf{Z_{AQ0}}$ is the incident special characteristic of the system for magnification $\mathbf{X_A}$; it defines incident special structures exactly as for $\mathbf{X}$ in the previous paper.

We turn now to the emergent segments of the special rays. Here the mathematics is a bit more complicated. From Equation 6

$$\mathbf{y}_0 = \mathbf{X}_\mathbf{A}^{-1}\mathbf{y}\,. \tag{14}$$

Substituting into Equation 2 and solving we obtain

$$\boldsymbol{\alpha}_0 = \mathbf{D}^{-1}\boldsymbol{\alpha} - \mathbf{D}^{-1}\mathbf{C}\mathbf{X}_\mathbf{A}^{-1}\mathbf{y}\,. \tag{15}$$

Substitution from Equations 4, 14 and 15 into Equation 1 results in

$$\left(\mathbf{I} - \left(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}\right)\mathbf{X}_\mathbf{A}^{-1}\right)\mathbf{y} = n\,\mathbf{B}\mathbf{D}^{-1}\mathbf{a}\,. \tag{16}$$

Now $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is the Schur complement of $\mathbf{A}$ in matrix $\mathbf{S}$. Because $\mathbf{S}$ is symplectic it reduces simply to $\mathbf{D}^{-\mathrm{T}}$ (see Equation 39 of a previous paper[5]), $\mathbf{D}^{-\mathrm{T}}$ being an abbreviation for the transpose of the inverse of $\mathbf{D}$. Hence

$$\left(\mathbf{I} - \mathbf{D}^{-\mathrm{T}}\mathbf{X}_\mathbf{A}^{-1}\right)\mathbf{y} = n\,\mathbf{B}\mathbf{D}^{-1}\mathbf{a}\,. \tag{17}$$

Substituting from Equation 11 and solving leads to

$$\mathbf{y}_Q = \left(\mathbf{I}z_Q + n\left(\mathbf{I} - \mathbf{D}^{-\mathrm{T}}\mathbf{X}_\mathbf{A}^{-1}\right)^{-1}\mathbf{B}\mathbf{D}^{-1}\right)\mathbf{a}. \tag{18}$$

But $\mathbf{B}\mathbf{D}^{-1}$ is a symmetric product[6], so we can replace it by its transpose and apply Equation 1 of the accompanying paper[6]. After a little rearrangement one obtains

$$\mathbf{y}_Q = \left(\mathbf{I}z_Q - \mathbf{Z}_{\mathbf{A}Q}\right)\mathbf{a} \tag{19}$$

where

$$\mathbf{Z}_{\mathrm{AQ}} = -n\left(\mathbf{D}^{\mathrm{T}} - \mathbf{X}_{\mathbf{A}}^{-1}\right)^{-1}\mathbf{B}^{\mathrm{T}} \tag{20}$$

is the emergent special characteristic of the system for magnification $\mathbf{X}_{\mathbf{A}}$. Equations 19 and 20 are the counterparts for $\mathbf{A}$ of Equations 42 and 41 for $\mathbf{D}$. $\mathbf{Z}_{\mathrm{AQ}}$ defines the location and nature of the emergent special structures.

Special rays with generalized magnifications $\mathbf{X}_{\mathbf{B}}$ and $\mathbf{X}_{\mathbf{C}}$ are handled in a similar fashion. One applies results for the Schur products for $\mathbf{B}$ and $\mathbf{C}$ (Equations 40 and 41 of the previous paper[6]) and the fact that certain products[6] are symmetric. For magnification $\mathbf{X}_{\mathbf{D}}$ the results are as determined before[3].

In each of the four cases we obtain

$$\mathbf{y}_{\mathrm{Q0}} = \left(\mathbf{I}z_{\mathrm{Q0}} - \mathbf{Z}_{\mathrm{Q0}}\right)\mathbf{a}_0 \tag{21}$$

for the intersection of the special rays with transverse plane T_{Q0}, the incident characteristic $\mathbf{Z}_{\mathrm{Q0}}$ being listed in Table 1. Similarly one obtains

$$\mathbf{y}_{\mathrm{Q}} = \left(\mathbf{I}z_{\mathrm{Q}} - \mathbf{Z}_{\mathrm{Q}}\right)\mathbf{a} \tag{22}$$

for the intersection of the special rays with T_{Q}, the emergent characteristic $\mathbf{Z}_{\mathrm{Q}}$ also being listed in Table 1.

Relationships among the generalized magnifications

Consider the special rays of generalized magnification $\mathbf{X}_{\mathbf{A}}$; they have incident characteristic $\mathbf{Z}_{\mathrm{AQ0}}$. Also consider the special rays with generalized magnification $\mathbf{X}_{\mathbf{B}}$ and incident characteristic $\mathbf{Z}_{\mathrm{BQ0}}$. It is apparent from Equation 21 that the two sets of special rays are in fact the same set if $\mathbf{Z}_{\mathrm{AQ0}} = \mathbf{Z}_{\mathrm{BQ0}}$. Since the special structures are defined by the special rays the special structures are the same as well. The same argument applies in all four cases. Thus different magnifications $\mathbf{X}_{\mathbf{A}}$, $\mathbf{X}_{\mathbf{B}}$, $\mathbf{X}_{\mathbf{C}}$ and $\mathbf{X}_{\mathbf{D}}$ imply the same special rays and special structures if

$$\mathbf{Z}_{\mathrm{AQ0}} = \mathbf{Z}_{\mathrm{BQ0}} = \mathbf{Z}_{\mathrm{CQ0}} = \mathbf{Z}_{\mathrm{DQ0}}. \tag{23}$$

In other words for a particular set of special rays the magnifications are all related by

$$\begin{aligned}(\mathbf{A} - \mathbf{X}_{\mathbf{A}})^{-1}\mathbf{B} &= \mathbf{A}^{-1}(\mathbf{B} - \mathbf{X}_{\mathbf{B}}) \\ &= (\mathbf{C} - \mathbf{X}_{\mathbf{C}})^{-1}\mathbf{D} = \mathbf{C}^{-1}(\mathbf{D} - \mathbf{X}_{\mathbf{D}}).\end{aligned} \tag{24}$$

By the same argument but based on the emergent segments one finds the magnifications also related by

$$\begin{aligned}\left(\mathbf{D}^{\mathrm{T}} - \mathbf{X}_{\mathbf{A}}^{-1}\right)^{-1}\mathbf{B}^{\mathrm{T}} &= \left(\mathbf{C}^{\mathrm{T}} + \mathbf{X}_{\mathbf{A}}^{-1}\right)^{-1}\mathrm{A}^{\mathrm{T}} \\ &= \mathbf{D}^{-\mathrm{T}}\left(\mathbf{B}^{\mathrm{T}} + \mathbf{X}_{\mathbf{C}}^{-1}\right) = \mathbf{C}^{-\mathrm{T}}\left(\mathrm{A}^{\mathrm{T}} - \mathbf{X}_{\mathbf{D}}^{-1}\right).\end{aligned} \tag{25}$$

From Equations 24 one finds that

$$\mathbf{X}_{\mathbf{A}} = \mathbf{A} - \mathbf{B}(\mathbf{D} - \mathbf{X}_{\mathbf{D}})^{-1}\mathbf{C}, \tag{26}$$

$$\mathbf{X}_{\mathbf{B}} = \mathbf{B} - \mathbf{A}\mathbf{C}^{-1}(\mathbf{D} - \mathbf{X}_{\mathbf{D}}), \tag{27}$$

$$\mathbf{X}_{\mathbf{C}} = \mathbf{C} - \mathbf{D}(\mathbf{D} - \mathbf{X}_{\mathbf{D}})^{-1}\mathbf{C}\ . \tag{28}$$

Thus for any given magnification $\mathbf{X}_{\mathbf{D}}$ for a set of special rays these equations give the other magnifications $\mathbf{X}_{\mathbf{A}}$, $\mathbf{X}_{\mathbf{B}}$ and $\mathbf{X}_{\mathbf{C}}$. An alternative set of equations can be obtained from Equations 25:

$$\mathbf{X}_{\mathbf{A}}^{-1} = \mathbf{D}^{\mathrm{T}} - \mathbf{B}^{\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}} - \mathbf{X}_{\mathbf{D}}^{-1}\right)^{-1}\mathbf{C}^{\mathrm{T}}, \tag{29}$$

Table 1: Incident $\mathbf{Z}_{\mathrm{Q0}}$ and emergent $\mathbf{Z}_{\mathrm{Q}}$ special characteristics for an optical system for the four generalized magnifications.

$\mathbf{Z}_{\mathrm{Q0}}$	$\mathbf{Z}_{\mathrm{Q}}$
$\mathbf{Z}_{\mathrm{AQ0}} = n_0(\mathbf{A} - \mathbf{X}_{\mathbf{A}})^{-1}\mathbf{B}$	$\mathbf{Z}_{\mathrm{AQ}} = -n\left(\mathbf{D}^{\mathrm{T}} - \mathbf{X}_{\mathbf{A}}^{-1}\right)^{-1}\mathbf{B}^{\mathrm{T}}$
$\mathbf{Z}_{\mathrm{BQ0}} = n_0\mathbf{A}^{-1}(\mathbf{B} - \mathbf{X}_{\mathbf{B}})$	$\mathbf{Z}_{\mathrm{BQ}} = -n\left(\mathbf{C}^{\mathrm{T}} + \mathbf{X}_{\mathbf{B}}^{-1}\right)^{-1}\mathbf{A}^{\mathrm{T}}$
$\mathbf{Z}_{\mathrm{CQ0}} = n_0(\mathbf{C} - \mathbf{X}_{\mathbf{C}})^{-1}\mathbf{D}$	$\mathbf{Z}_{\mathrm{CQ}} = -n\mathbf{D}^{-\mathrm{T}}\left(\mathbf{B}^{\mathrm{T}} + \mathbf{X}_{\mathbf{C}}^{-1}\right)$
$\mathbf{Z}_{\mathrm{DQ0}} = n_0\mathbf{C}^{-1}(\mathbf{D} - \mathbf{X}_{\mathbf{D}})$	$\mathbf{Z}_{\mathrm{DQ}} = -n\mathbf{C}^{-\mathrm{T}}\left(\mathbf{A}^{\mathrm{T}} - \mathbf{X}_{\mathbf{D}}^{-1}\right)$

$$\mathbf{X}_\mathbf{B}^{-1} = -\mathbf{C}^\mathrm{T} + \mathbf{A}^\mathrm{T}\left(\mathbf{A}^\mathrm{T} - \mathbf{X}_\mathbf{D}^{-1}\right)^{-1}\mathbf{C}^\mathrm{T}, \tag{30}$$

$$\mathbf{X}_\mathbf{C}^{-1} = -\mathbf{B}^\mathrm{T} + \mathbf{D}^\mathrm{T}\mathbf{C}^{-\mathrm{T}}\left(\mathbf{A}^\mathrm{T} - \mathbf{X}_\mathbf{D}^{-1}\right). \tag{31}$$

From Equations 26 to 28 one obtains

$$\mathbf{X}_\mathbf{D} = \mathbf{D} - \mathbf{C}(\mathbf{A} - \mathbf{X}_\mathbf{A})^{-1}\mathbf{B}, \tag{32}$$

$$\mathbf{X}_\mathbf{D} = \mathbf{D} - \mathbf{C}\mathbf{A}^{-1}(\mathbf{B} - \mathbf{X}_\mathbf{B}), \tag{33}$$

$$\mathbf{X}_\mathbf{D} = \mathbf{D} - \mathbf{C}(\mathbf{C} - \mathbf{X}_\mathbf{C})^{-1}\mathbf{D}. \tag{34}$$

We mention in passing that a number of identities are implied here. For example Equations 26 and 29 show that

$$\left(\mathbf{A} - \mathbf{B}(\mathbf{D} - \mathbf{X})^{-1}\mathbf{C}\right)^{-1} = \mathbf{D}^\mathrm{T} - \mathbf{B}^\mathrm{T}\left(\mathbf{A}^\mathrm{T} - \mathbf{X}^{-1}\right)\mathbf{C}^{-\mathrm{T}} \tag{35}$$

for any matrix $\mathbf{X}$ and for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ the submatrices of a symplectic matrix $\begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}$. Of course here, as elsewhere in this paper, the relationship depends on the existence of particular inverses.

Generalized magnifications for cardinal rays and structures

For principal rays and structures $\mathbf{X}_\mathbf{D} = \mathbf{I}$.[3] Substituting into Equation 26 to 28 we obtain their other generalized magnifications:

$$\mathbf{X}_\mathbf{A} = \mathbf{A} - \mathbf{B}(\mathbf{D} - \mathbf{I})^{-1}\mathbf{C}, \tag{36}$$

$$\mathbf{X}_\mathbf{B} = \mathbf{B} - \mathbf{A}\mathbf{C}^{-1}(\mathbf{D} - \mathbf{I}), \tag{37}$$

$$\mathbf{X}_\mathbf{C} = \mathbf{C} - \mathbf{D}(\mathbf{D} - \mathbf{I})^{-1}\mathbf{C}. \tag{38}$$

Because of the Schur complement[3] Equation 36 can be written

$$\mathbf{X}_\mathbf{B} = -\mathbf{C}^{-\mathrm{T}} + \mathbf{A}\mathbf{C}^{-1}, \tag{39}$$

and because $\mathbf{A}\mathbf{C}^{-1}$ is symmetric this can be further reduced to

$$\mathbf{X}_\mathbf{B} = \mathbf{C}^{-\mathrm{T}}\left(\mathbf{A}^\mathrm{T} - \mathbf{I}\right). \tag{40}$$

Although $\mathbf{A}\mathbf{C}^{-1}$ is symmetric for all systems $\mathbf{C}$ can be asymmetric. It follows from Equation 39 that $\mathbf{X}_\mathbf{B}$ can be asymmetric. Though less obvious $\mathbf{X}_\mathbf{A}$ and $\mathbf{X}_\mathbf{C}$ can also be asymmetric for principal rays and structures.

Similarly for nodal rays and structures $\mathbf{X}_\mathbf{D} = \mathbf{I}n/n_0$.[3] Hence for them the generalized magnifications are

$$\mathbf{X}_\mathbf{A} = \mathbf{A} - \mathbf{B}(\mathbf{D} - \mathbf{I}n/n_0)^{-1}\mathbf{C}, \tag{41}$$

$$\mathbf{X}_\mathbf{B} = \mathbf{B} - \mathbf{A}\mathbf{C}^{-1}(\mathbf{D} - \mathbf{I}n/n_0), \tag{42}$$

$$\mathbf{X}_\mathbf{C} = \mathbf{C} - \mathbf{D}(\mathbf{D} - \mathbf{I}n/n_0)^{-1}\mathbf{C}. \tag{43}$$

For similar reasons they too can be asymmetric. They are calculated for a particular model eye in the Appendix.

Incident focal rays and structures have $\mathbf{X}_\mathbf{D} = \mathbf{O}$.[3] Substituting this into Equations 26 to 28 and recognizing the Schur complements one obtains

$$\mathbf{X}_\mathbf{A} = \mathbf{D}^{-\mathrm{T}}, \tag{44}$$

$$\mathbf{X}_\mathbf{B} = -\mathbf{C}^{-\mathrm{T}}, \tag{45}$$

$$\mathbf{X}_\mathbf{C} = \mathbf{O}. \tag{46}$$

For emergent focal rays and structures $\mathbf{X}_\mathbf{D} = \mathbf{I}\infty$.[3] Recognizing Schur complements and symmetry we obtain

$$\mathbf{X}_\mathbf{A} = \mathbf{A}, \tag{47}$$

$$\mathbf{X}_\mathbf{B}^{-1} = \mathbf{O}, \tag{48}$$

$$\mathbf{X}_\mathbf{C} = \mathbf{C}. \tag{49}$$

Special rays and structures with particular $\mathbf{X}_\mathbf{A}$

A counterpart of principal rays and structures (with $\mathbf{X}_\mathbf{D} = \mathbf{I}$) would be the set of special rays and structures for which $\mathbf{X}_\mathbf{A} = \mathbf{I}$; each ray in the set emerges relative to the optical axis at the same position that it enters. From Equation 32 one observes that

$$\mathbf{X}_\mathbf{D} = \mathbf{D} - \mathbf{C}(\mathbf{A} - \mathbf{I})^{-1}\mathbf{B}. \tag{50}$$

More interesting, perhaps, are special rays and structures for which $\mathbf{X}_\mathbf{A} = \mathbf{O}$. They represent all the rays that emerge from the system on the optical axis. From Equation 32 we see that

$$\mathbf{X}_\mathbf{D} = \mathbf{A}^{-\mathrm{T}}. \tag{51}$$

From Table 1 we see that the emergent special characteristic $\mathbf{Z_{DQ}}$ is null which is in agreement with the fact that the emergent special structure is a point on the retina in the case of an eye. The incident special characteristic is

$$\mathbf{Z_{DQ0}} = n_0\mathbf{C}^{-1}\left(\mathbf{D} - \mathbf{A}^{-\mathrm{T}}\right). \tag{52}$$

Using the fact that $\mathbf{A}^{-\mathrm{T}}$ is the Schur complement of $\mathbf{D}$ we find that

$$\mathbf{Z_{DQ0}} = n_0\mathbf{A}^{-1}\mathbf{B} \tag{53}$$

which is symmetric. This in fact defines the incident special structure (an interval of Sturm) conjugate to the retina. Because the corneal-plane refractive compensation[7] is $\mathbf{F}_0 = \mathbf{B}^{-1}\mathbf{A}$ we can write

$$\mathbf{Z_{DQ0}} = n_0\mathbf{F}_0^{-1}. \tag{54}$$

Concluding remarks

Corresponding to each submatrix of **A**, **B**, **C** and **D** of the transference **S** is a generalized magnification, $\mathbf{X_A}$, $\mathbf{X_B}$, $\mathbf{X_C}$ and $\mathbf{X_D}$. Associated with each are an incident and an emergent special characteristic as listed in Table 1. Specifying a particular generalized magnification defines a set of special rays; the incident segments of the rays are defined by Equation 21 and the emergent segments by Equation 22. Associated with those special rays may be special points or lines (like nodal, principal and focal points and lines). Their locations and natures are given by the eigenstructures of the special characteristics exactly as described before[3] for generalized magnification $\mathbf{X_D}$.

While our interests here have been primarily theoretical, and more particularly mathematical generalization and completeness, we imagine that $\mathbf{X_A}$, $\mathbf{X_B}$ and $\mathbf{X_C}$ may have relevance where position on the cornea and retina are of interest.

In one aspect this paper has not aimed at completeness. There are many situations in which particular matrices are singular. Then, of course, equations involving their inverses do not hold. It seems likely that they would be of little or no practical interest and have, therefore, been given no more than passing mention.

Acknowledgements

I thank the National research Foundation for support.

References

1. Gauss CF. Dioptrische Untersuchungen. *Abhandlungen der Königlichen Gesellschaft der Wissenschaften in Göttingen (Mathematische Klasse)* 1843 **1** 1-34.
2. Listing JB. Mathematische Discussion des Ganges der Lichtstrahlen im Auge. In: *Handwörterbuch der Physiologie mit Rücksicht auf physiologische Pathologie*. Vol. 4, edited by Wagner, R., Braunschweig, Vieweg, 1853, 451-504.
3. Harris WF. Cardinal points and generalizations. *Ophthal Physiol Opt* 2010 **30** 391-401.
4. Harris WF. Nodes and nodal points and lines in eyes and other optical systems. *Ophthal Physiol Opt* 2010 **30** 24-42.
5. Maxwell JC. Address to the Mathematical and Physical Sections of the British Association. B*ritish Association for the Advancement of Science* 1870 Report 40. Republished in: *The Scientific Papers of James Clerk Maxwell, Volume II*. Cambridge Univ Press, Cambridge, 1890, 215-229. Republished by Dover, Mineola NY, 2003.
6. Harris WF. Symplecticity and relationships among the fundamental properties in linear optics. *S Afr Optom* 2010 **69** 3-13.
7. Harris WF. A unified paraxial approach to astigmatic optics. *Optom Vis Sci* 1999 **76** 480-499.

Appendix

Generalized magnifications of nodal rays for a model eye

As a numerical example we calculate generalized magnifications for nodal rays in the case of the model eye defined before[3, 4]. From Equations 40 to 42 we obtain

$$\mathbf{X_A} = \begin{pmatrix} -2.576 & 0.035 \\ 0.026 & -2.353 \end{pmatrix},$$

$$\mathbf{X_B} = \begin{pmatrix} 17.074 & -0.064 \\ -0.059 & 16.594 \end{pmatrix} \text{mm}$$

and

$$\mathbf{X_C} = \begin{pmatrix} -201.107 & 2.021 \\ 1.515 & -189.051 \end{pmatrix} \text{D}$$

respectively. All three are strictly asymmetric although not very different from scalar matrices. We see that, for this particular eye, nodal rays undergo a magnification in transverse position from cornea to retina of about -2.5. There is a cross-magnification of about 17 mm for reduced inclination at the cornea to transverse position at the retina (about 0.17 mm per centiradian) and a cross-magnification of about -200 D, that is, about -0.2 radians per millimetre, of transverse position at the cornea to reduced inclination at the retina.