

Dependence of the transference of a reduced eye on frequency of light*

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Abstract

In Gaussian optics the transference is a matrix that is a complete representation of the effects of the system on a ray traversing it. Almost all of the familiar optical properties of the system, such as refractive error and power of the system, can be calculated from the transference. Because of the central importance of the transference it is useful to have some idea of how it depends on the frequency of light. This paper examines the simplest model eye, the reduced eye. The dependence of the transference is calculated in terms of both frequency and

wavelength of light and both dependencies are displayed graphically. The principal matrix logarithms are also calculated and displayed graphically. Chromatic difference in refractive compensation, power and ametropia are obtained for the reduced eye from the transferences. (*S Afr Optom* 2011 70(4) 149-155)

Key Words: Transference, frequency, wavelength, Emsley's reduced eye, transformed transference, chromatic difference in: refractive compensation, power and ametropia.

Introduction

A transference is a matrix that represents the linear optical properties of an optical system, such as the eye. In Gaussian optics the transference is a complete representation of the effects of the system on a ray traversing it. Most of the optical properties of the system, such as refractive compensation¹, back- and front-vertex power², locations of the cardinal points^{3, 4} and power of the system⁵ can be calculated from the system's transference. Because of the central importance of the transference it is useful to have some idea of how it depends on the frequency of the light traversing it. We take a look at the simplest model eye, the reduced eye.

The dependence of the fundamental first-order properties, calculated in terms of both frequency and wavelength of light, will be represented graphically across the visible light spectrum. Further, the

dependence of the transformed transference will be represented graphically. Formulae for the calculation of chromatic difference in corneal-plane refractive-compensation of the reduced eye and chromatic difference in power and ametropia are derived.

Emsley's reduced eye

The advantage of the reduced eye (see Figure 1) is its simplicity. The reduced eye has a single stigmatic refracting surface of radius of curvature r and a homogenous gap of length z . Emsley⁶ designed his reduced eye to match certain measurements of the Gullstrand-Emsley schematic eye. He placed specific emphasis on the power of 60 D and chose the refractive index in the reduced eye to be the same as that of water, namely 4/3. He took the index of air to be 1. These numbers imply a radius of curvature r of 50/9 mm (or 1/180 m) and a length z of 200/9 mm (or 1/45

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m)^{6,7}. In this paper we will make use of a reduced eye with the same r and z as Emsley's eye and with an index n dependant on the frequency of light.

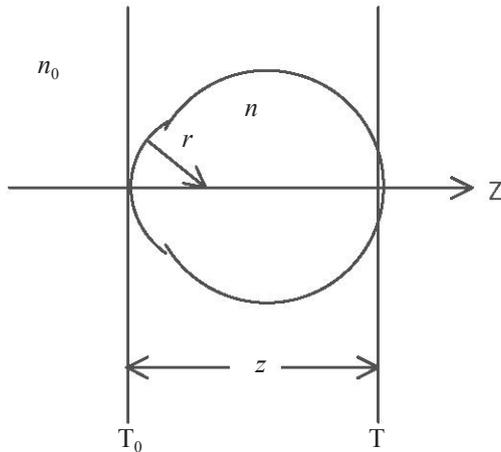


Figure 1. The reduced eye as a defined optical system. The length is z , the radius of curvature of the refracting surface is r , the refractive index outside the system is n_0 and inside the system is n . The optical system has a longitudinal axis, Z and is bound by an entrance plane T_0 immediately in front of the refracting surface and an exit plane T immediately in front of the retinal surface.

Linear optics

An optical system is bound by an entrance plane T_0 and an exit plane T and has a longitudinal axis Z . The transference \mathbf{T} of a stigmatic, untilted, centred Gaussian optical system is represented by⁸

$$\mathbf{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

where A the dilation, B the disjugacy, C the divergence and D the divarication are the fundamental first-order optical properties of the system^{1, 8, 9}. The power F is given simply by^{5, 8}

$$F = -C \tag{2}$$

When the system is an eye the dilation A can be considered to be the ametropia⁸ of the eye. In particular when the eye is emmetropic

$$A = 0 \tag{3}$$

The corneal-plane refractive compensation is given by^{1, 8}

$$F_0 = B^{-1}A \tag{4}$$

The transferences of the two elementary optical systems^{10, 11} are

$$\mathbf{T}_\xi = \begin{pmatrix} 1 & \frac{z}{n} \\ 0 & 1 \end{pmatrix} \tag{5}$$

for a homogenous gap and

$$\mathbf{T}_K = \begin{pmatrix} 1 & 0 \\ -\frac{n-n_0}{r} & 1 \end{pmatrix} \tag{6}$$

for a refracting surface. To calculate the transference of the reduced eye one multiplies in reverse¹² as follows

$$\mathbf{T} = \mathbf{T}_\xi \mathbf{T}_K \tag{7}$$

Substituting Equations 5 and 6 into Equation 7 and multiplying one obtains

$$\mathbf{T} = \begin{pmatrix} 1 - \frac{z}{n} \left(\frac{n-n_0}{r} \right) & \frac{z}{n} \\ -\frac{n-n_0}{r} & 1 \end{pmatrix} \tag{8}$$

Substituting the values for Emsley's reduced eye into Equation 8 one obtains

$$\mathbf{T} = \begin{pmatrix} 0 & \frac{50}{3} \text{ mm} \\ -0.060 \text{ kD} & 1 \end{pmatrix} \tag{9}$$

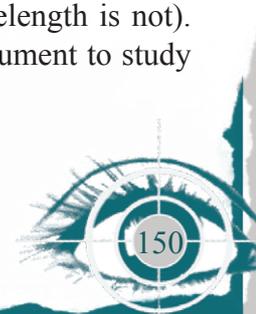
It is immediately apparent from Equations 3 and 9 that the eye is emmetropic and has a power of 60 D.

Frequency or wavelength?

The fundamental relationship between frequency (ν) and vacuum wavelength (λ) is given by

$$c_0 = \nu \lambda \tag{10}$$

where light traveling in a vacuum has a speed $c_0 = 299\,792\,458 \text{ m.s}^{-1}$ as defined by the 17th General Conference on Weights and Measures in November 1983. Pease and Barbeito¹³ look at the relationship between frequency and wavelength for a number of studies involving chromatic aberration and conclude that results using frequency or wavenumber (the inverse of wavelength) are "nearly perfectly linear"^{13, 14} in contrast to those using wavelength. They cite several reasons to support using frequency rather than wavelength (perhaps the most important being that frequency is independent of the medium whereas wavelength is not). These reasons make a compelling argument to study



the dependence of the transference on the frequency of light rather than its dependence on wavelength. We will compare the dependence of the transference on both frequency and wavelength.

Visible light colour spectrum

The spectrum with wavelengths in vacuum is between 400 and 700 nm and represents the range over which human spectral sensitivity varies between 1 and 100%.¹⁴⁻¹⁶ This represents frequencies between 428.3 THz and 749.5 THz, approximately. Six coloured reference points are shown on each of the graphs below. The four colours red-orange, yellow, green and blue represent the peak vacuum wavelengths where each colour is considered “pure”¹⁷ while the deep-red and violet-blue represent the spectral range end-points. The frequencies and wavelengths of the six colours are detailed in Table 1.

Transference as a function of refractive index

The transference of the reduced eye in Equation 8 shows that the variable affected by different frequencies of light is the refractive index *n*. Entering the radius of curvature and length of Emsley’s reduced eye we obtain its transference as a function of the refractive indices:

$$T = \begin{pmatrix} \frac{4n_0 - 3}{n} & \frac{200}{9n} \text{ mm} \\ -\frac{9}{50}(n - n_0) \text{ kD} & 1 \end{pmatrix} \quad (11)$$

Refractive index of the Reduced Eye

Thibos *et al*¹⁵ represent the refractive index of the reduced eye as a function of wavelength as follows

$$n = a + \frac{b}{\lambda - c} \quad (12)$$

where a=1.320535, b=4.685 nm and c=214.102 nm. The formula is based on Cornu’s formula for refractive index of water and constants were derived from clinical experimentation on real eyes. Using this formula, Thibos *et al*¹⁵ showed that the refractive index of the body of the reduced eye changes more rapidly with wavelength than a reduced eye filled with water. The predictions for longitudinal chromatic aberration using this formula more closely approximate experimental data than Emsley’s reduced eye filled with water. The refractive indices for our six reference points were calculated using Equation 12 and are given in Table 1.

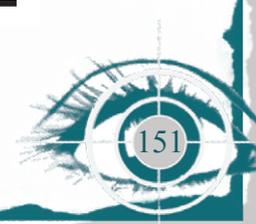
Refractive index of air

The refractive index of air differs only very slightly from that of a vacuum and for most optometric calculations one can put *n*₀=1. A number of equations, for example Cauchy’s dispersion formula¹⁸ and Ciddor’s equations¹⁹, are available for calculating the refractive index of air. Cauchy’s formula is expressed in terms of wavelength whereas Ciddor’s equations are expressed in terms of wavenumber. Cauchy’s dispersion formula¹⁸ is

$$(n_0 - 1)10^7 = p + \frac{q}{\lambda^2} + \frac{t}{\lambda^4} \quad (13)$$

Table 1. The colours of the six reference points and their frequencies, vacuum wavelengths, refractive indices for the reduced eye, ametropias, powers for the reduced eye and refractive compensations are provided. The last row shows the chromatic difference across the spectrum 428.3 to 749.5 THz.

Colour	Frequency in THz	Vacuum wave-length in nm	Refractive index	Ametropia	Power of reduced eye in dioptres	Refractive compensation in dioptres
Deep red	428.27	700	1.3302	0.0071	59.4318	0.4261
Red	475.86	630	1.3318	0.0035	59.7240	0.2070
Yellow	516.88	580	1.3333	0.0000	60.0010	0.0008
Green	576.52	520	1.3359	-0.0057	60.4531	-0.3398
Blue	631.14	475	1.3385	-0.0116	60.9286	-0.6964
Violet-blue	749.48	400	1.3457	-0.0277	62.2327	-1.6745
Chromatic difference in:			Δn	ΔA	ΔF	ΔF_0
			= 0.0156	= -0.0348	= 2.8008	= -2.1006



where $p=2726.43$, $q=12.288 \times 10^6 \text{ nm}^2$ and $t=355.5 \times 10^9 \text{ nm}^4$ for dry air at temperature 15°C and pressure 101 kPa .

When Cauchy's dispersion formula is used for n_0 one obtains results that differ insignificantly from those for $n_0=1$. In particular for dilation the difference is less than 0.00083 across the entire visible light spectrum. For divergence this difference is 0.05 D . This results in a very slight upward shift in the graphs for A and C in Figure 2, however, the curvatures are unaffected.

Calculation shows that the reduced eye is emmetropic at the frequency 517 THz (580 nm , yellow) when we use $n_0=1$ but is emmetropic at the frequency 526 THz (570 nm , yellow-green) when the refractive index of air is calculated using Cauchy's formula.

Graphical representation of the fundamental properties

The properties are calculated for frequencies according to Equation 11 with $n_0=1$ and n given by Equation 12. The results of the calculations of each of the fundamental first-order optical properties are given in the accompanying graphs. Figure 2 represents each of the fundamental properties as functions of

frequency of light and Figure 3 as functions of wavelength. The six coloured reference points are shown by means of coloured diamonds. The small black dots on Figure 2 represent 10 equally spaced intervals of frequency of approximately 32.1 THz and the crosses on Figure 3 represent 10 equally spaced intervals of 30 nm wavelength.

We note that in Figure 2 the dots are more evenly spaced than the crosses in Figure 3. The dashed line represents the slope of the curve calculated using the least squares method. Each of A , B and C present as curves in both sets of graphs. D is a straight line at 1 as required by Equation 11. The curves are closer to straight lines in Figure 2 than in Figure 3; this provides some justification for preferring to think in terms of frequency rather than wavelength of light.

The transferences at the extremes of the visible spectrum (428.3 THz or 700 nm and 749.5 THz or 400 nm), are

$$\mathbf{T}_{428.3 \text{ THz}} = \begin{pmatrix} 0.0071 & 16.7062 \text{ mm} \\ -0.0594 \text{ kD} & 1 \end{pmatrix} \quad (14)$$

$$\mathbf{T}_{749.5 \text{ THz}} = \begin{pmatrix} -0.0277 & 16.5130 \text{ mm} \\ -0.0622 \text{ kD} & 1 \end{pmatrix} \quad (15)$$

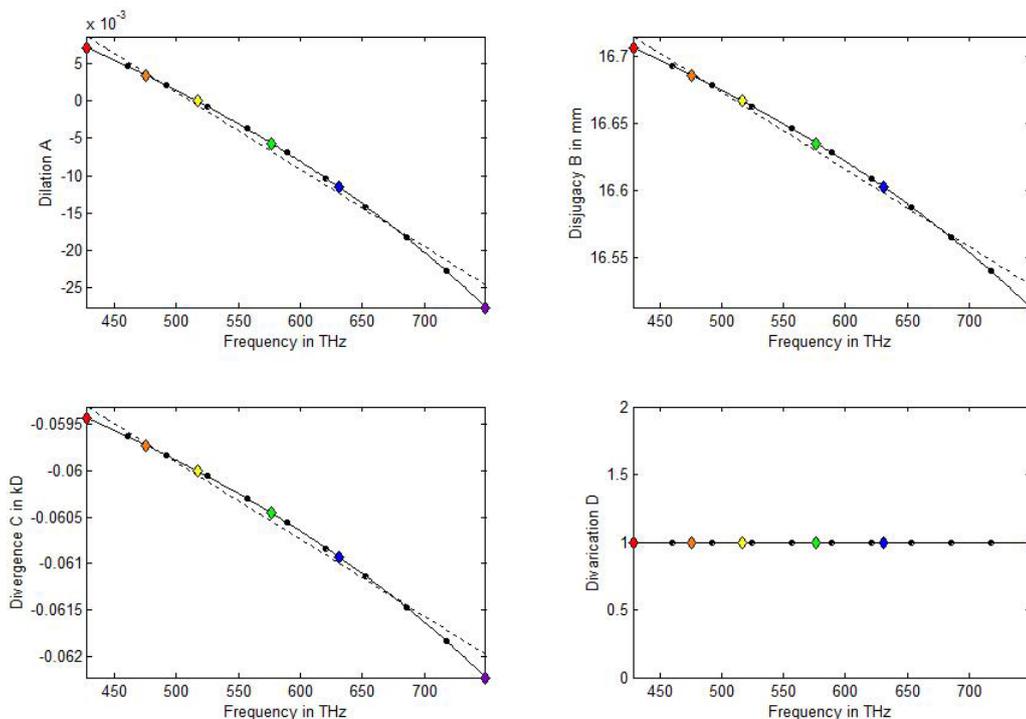


Figure 2. Sub-graphs A, B, C and D represent the four fundamental optical properties of the reduced eye as a function of frequency of light. The six coloured diamonds indicate six reference points as indicated in Table 1. The small black dots represent 10 equal intervals of 32.1 THz each. Each of A , B and C present as curves, while D is a horizontal straight line at 1.



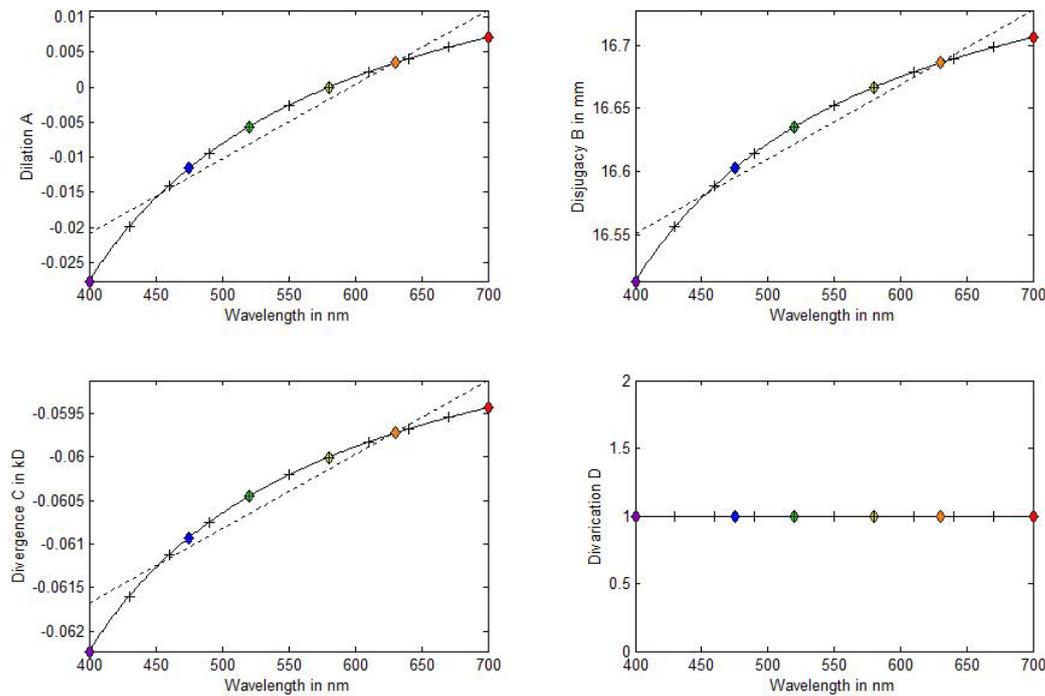


Figure 3. The four fundamental properties of the reduced eye as a function of wavelength. The black crosses represent 10 equal intervals of 30 nm each.

The transformed transference

By taking the principal matrix logarithm we convert each transference T into a Hamiltonian matrix^{8, 20}. We represent the transformed matrix⁸ by \hat{T} . Thus $\hat{T} = \text{Log}T$. (16)

In MATLAB the function used is *logm*. In terms of its entries we write the transformed transference⁸ as

$$\hat{T} = \begin{pmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{pmatrix}. \quad (17)$$

Because \hat{T} is Hamiltonian²⁰,

$$\hat{A} = -\hat{D}. \quad (18)$$

\hat{T} therefore has only three independent entries. This creates a 3-dimensional vector space which can be plotted on a 3-dimensional graph as done in Figure 4. In the figure \hat{A}, \hat{B} and \hat{C} are along three orthogonal axes, where \hat{A} (and \hat{D}) are unitless, \hat{B} is in millimetres and \hat{C} is in kilodiotres. The result is close to a straight line. The small black dots represent 20 equally spaced intervals of frequency of 16.1 THz.

Transforming Equations 14 and 15, we find

$$\hat{T}_{428.3\text{THz}} = \begin{pmatrix} -0.5994 & 20.1699 \text{ mm} \\ -0.0718 \text{ kD} & 0.5994 \end{pmatrix} \quad (19)$$

$$\hat{T}_{749.5\text{THz}} = \begin{pmatrix} -0.6251 & 20.0888 \text{ mm} \\ -0.0757 \text{ kD} & 0.06251 \end{pmatrix}. \quad (20)$$

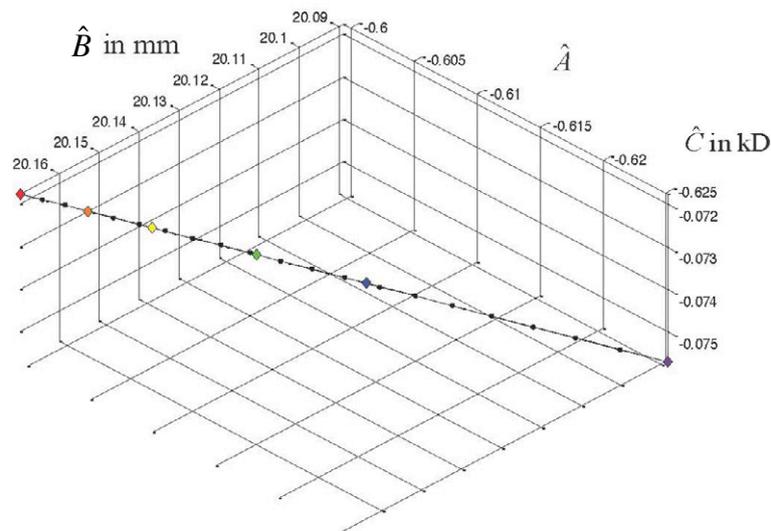


Figure 4. The transformed transference of the reduced eye in Hamiltonian space. The three axes represent \hat{A}, \hat{B} and \hat{C} . The black dots represent frequencies spaced at intervals of 16.06 THz. The six diamonds represent the six coloured reference points. (The azimuth of 45° and elevation of 125° were chosen to exaggerate any possible curvature.)



Some Derived Properties

We obtain here some derived properties of the reduced eye that are directly and simply obtainable from the transference. Using Equation 2 we calculate the power F of the reduced eye for a particular frequency of light and using Equation 4 we calculate the corneal-plane refractive compensation F_0 for the frequency. We read the ametropia A directly from the transference for the chosen frequency of light. Of greater interest is the chromatic difference in refractive compensation ΔF_0 (known by various terms including longitudinal chromatic aberration²¹, chromatic difference of refractive error²¹, chromatic difference in refraction¹⁴, and axial chromatic aberration²¹), chromatic difference in power (also known as chromatic difference of equivalent power¹⁴) and chromatic difference in ametropia across a specified spectrum of visible light to be studied.

Each of these derived properties in chromatic difference between two frequencies or wavelengths can be calculated in two ways: either directly from the two transferences or using the equations below. Equations 21, 23, 24 and 26 are general equations while Equations 22, 25 and 27 apply to the reduced eye.

To calculate the chromatic difference in corneal-plane refractive compensation across a specified spectrum we take values from the two transferences (Equation 1) and substitute them into Equation 4 as follows:

$$\Delta F_0 = B_2^{-1}A_2 - B_1^{-1}A_1 \quad (21)$$

Now substituting from Equation 8 and also substituting the values for z and r for the reduced eye, we find

$$\Delta F_0 = (-135 \text{ D})\Delta n \quad (22)$$

where Δn simplifies to

$$\Delta n = b \left(\frac{1}{\lambda_2 - c} - \frac{1}{\lambda_1 - c} \right) \quad (23)$$

and b and c are the same constants as for Equation 12.

To calculate chromatic difference in power we similarly take values from the transferences and substitute them into Equation 2 as follows

$$\Delta F = -(C_2 - C_1) \quad (24)$$

and substituting from Equation 8 and then substituting the value for r for the reduced eye we find

$$\Delta F = (180 \text{ D})\Delta n \quad (25)$$

The chromatic difference in ametropia is derived directly from the transferences and therefore

$$\Delta A = A_2 - A_1 \quad (26)$$

Substituting values for z and r we obtain

$$\Delta A = 4 \left(\frac{1}{n_2} - \frac{1}{n_1} \right) \quad (27)$$

where n_1 and n_2 are calculated from Equation 12.

The chromatic difference in corneal-plane refractive compensation of the reduced eye was calculated across the visible light spectrum 428.3 to 749.5 THz and is -2.1006 D (see Table 1). The chromatic difference in power is 2.8 D . The chromatic difference in ametropia is -0.0348 .

Conclusion

The transference of the reduced eye depends on the frequency of light. The accuracy of the calculations depends on the formula used to calculate the refractive index as a function of either frequency of light or wavelength. In this article we have used Equation 12.

Results are displayed graphically for the transferences both as a function of frequency and as a function of wavelength. Divergence D is constant while ametropia A , disjucacy B and divergence C exhibit curved lines. The graph for the transformed transference in Hamiltonian space is approximately a straight line.

Chromatic difference in refractive compensation, chromatic difference in power and chromatic difference in ametropia are calculated directly from the transference for the reduced eye.

Acknowledgements

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